Constraint Satisfaction

Ch. 6.1–6.4 (skip 6.3.3))

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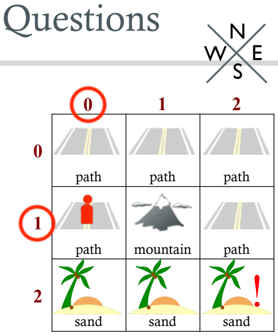
HW 2 Questions

- Do you have the latest version?
- Coords: **(row, column)**
  - Start at (1,0)
  - Arguments in:
  - ((start), (goal), [[array]])
- What three algorithms would you use?
  - Uninformed/Blind
  - Informed
  - **Local**



Today's Class

- What's a Constraint Satisfaction Problem (CSP)?
  - A.K.A., Constraint Processing / CSP paradigm
- How do we solve them?
  - Algorithms for CSPs
- Search Terminology

Constraint (n): A relation … between the values of one or more mathematical variables (e.g., x>3 is a constraint on x).

Constraint satisfaction assigns values to variables so that all constraints are true.

– *http://foldoc.org/constraint*

Constraint Satisfaction

- Con•straint /kənˈstrānt/, (noun):
  - Something that limits or restricts someone or something.[1]
  - A relation … between the values of one or more mathematical variables (e.g., x>3 is a constraint on x).[2]
  - Assigns values to variables so that all constraints are true.[2]
- In search, constraints exist on?
- General Idea
  - View a problem as a **set of variables**
  - To which we have to assign **values**
  - That satisfy a number of (problem-specific) **constraints**

[1] Merriam-Webster online.
[2] The Free Online Computing Dictionary.

Overview

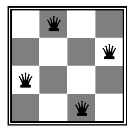
- **Constraint satisfaction**: a problem-solving paradigm
- Constraint programming, constraint satisfaction problems (**CSPs**), constraint logic programming…
- Algorithms for CSPs
  - Backtracking (systematic search)
  - Constraint propagation (k-consistency)
  - Variable and value ordering heuristics
  - **Backjumping and dependency-directed backtracking**

Search Vocabulary

- We've talked about caring about *goals* (end states) vs. *paths*
- These correspond to…
  - **Planning**: finding sequences of actions
    - Paths have various costs, depths
    - Heuristics to guide, frontier to keep backup possibilities
    - Examples: chess moves; 8-puzzle; homework 2
  - **Identification**: assignments to variables representing unknowns
    - The goal itself is important, not the path
    - Examples: Sudoku; map coloring; N queens
- CSPs are specialized for identification problems

Slightly Less Informal Definition of CSP

- **CSP** = Constraint Satisfaction Problem
- Given:
  1. A finite set of **variables**
  2. Each with a **domain** of possible values they can take (often finite)
  3. A set of **constraints** that limit the values the variables can take on
- **Solution**: an assignment of values to variables that satisfies all constraints.



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CSP Applications

- Decide if a solution exists
- Find some solution
- Find all solutions
- Find the "best solution"
  - According to some metric (objective function)
  - Does that mean "optimal"?

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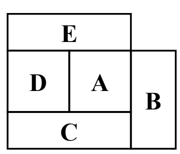
Informal Example: Map Coloring

- Given a 2D map, it is always possible to color it using three colors (we'll use red, green, blue)

  Such that:

- No two adjacent regions are the same color
- **Start thinking:** What are the values, variables, constraints?



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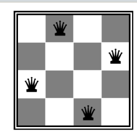
Slightly Less Informal

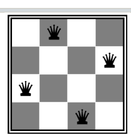
- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - **State** is defined by variables X_i with values from a domain D
    - Sometimes D depends on i
- Goal test is a **set of constraints** specifying allowable combinations of values for subsets of variables



Example: N-Queens (1)

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints:

$$\forall i,j,k \;\; (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\sum_{i,j} X_{ij} = N$$



Example: N-Queens (2)

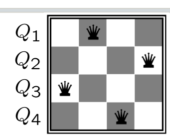
- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

Implicit: $\forall i, j$ $\texttt{non-threatening}(Q_i, Q_j)$

-or-

Explicit: $(Q_1, Q_2) \in \{(1,3),(1,4),\ldots\}$

...



Example: SATisfiability

- Given a set of propositions containing variables, find an assignment of the variables to {false, true} that satisfies them. Special case!
- For example, the clauses:
  - (A ∨ B ∨ ¬C) ∧ (¬A ∨ D)
  - (equivalent to (C → A) ∨ (B ∧ D → A))

  are satisfied by
  A = false
  B = true
  C = false
  D = false

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Real-World Problems

- Scheduling
- Temporal reasoning
- Building design
- Planning
- Optimization/ satisfaction
- Vision
- Graph layout
- Network management
- Natural language processing
- Molecular biology / genomics
- VLSI design

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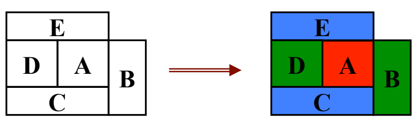
Map Coloring II

- Variables:
- Domains:
- Constraints:
- One solution: A=red, B=green, C=blue, D=green, E=blue



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Formal Definition: Constraint Network (CN)

A **constraint network** (CN) consists of

- A set of variables $X = \{x_1, x_2, \dots x_n\}$
  - Each with an associated domain of values $\{d_1, d_2, \dots d_n\}$.
  - The domains are typically finite
- A set of constraints $\{c_1, c_2 \dots c_m\}$ where
  - Each constraint defines a **predicate,** which is a **relation** over some subset of X.
  - E.g., c_i involves variables $\{X_{i1}, X_{i2}, \dots X_{ik}\}$ and defines the relation $R_i \subseteq D_{i1} \times D_{i2} \times \dots D_{ik}$

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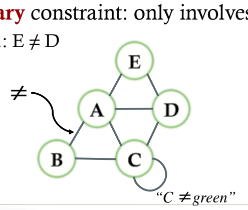
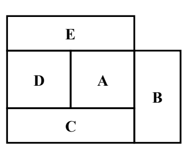
Constraint Restrictions

- **Unary** constraint: only involves one variable
  - e.g.: C can't be green.
- **Binary** constraint: only involves two variables
  - e.g.: E ≠ D





Formal Definition of a CN (cont.)

- An **instantiation** is an assignment of a value $d_x \in D$ to some subset of variables S.
  - Any assignment of values to variables
  - Ex: $\underline{Q_2 = \{2,3\} \wedge Q_3 = \{1,1\}}$ **instantiates** Q_2 and Q_3
- An instantiation is **legal** iff it does not violate any constraints
- A **solution** is an instantiation of all variables
  - A **correct solution** is a **legal** instantiation of all variables

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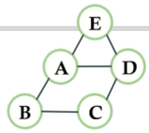
Typical Tasks for CSP



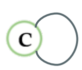
- Solutions:
  - Does a solution exist?
  - Find one solution
  - Find all solutions
  - Given a *partial instantiation*, can we do these?
- Transform the CN into an equivalent CN that is easier to solve

Binary CSP

- **Binary CSP**: all constraints are binary or unary
- Can convert a non-binary CSP → binary CSP by:
  - Introducing additional variables
  - One variable per constraint
  - One binary constraint for each pair of original constraints that share variables
- "Dual graph construction"

Binary CSPs: Why?

- Can always represent a binary CSP as a **constraint graph** with:
  - A **node** for each variable
  - An **arc** between two nodes iff there is a constraint on the two variables
  - Unary constraint appears as a self-referential arc



"C can't be green"

Example: Sudoku

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

- Variables
  - $v_{i,j}$ is the value in the j^{th} cell of the i^{th} row
- Domains
  - $D_{i,j} = D = \{1, 2, 3, 4\}$
- Blocks:
  - $B_1 = \{11, 12, 21, 22\}, \ldots, B_4 = \{33, 34, 43, 44\}$

Running Example: Sudoku

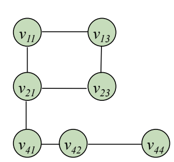
v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

- Constraints (implicit/intensional)
  - $C^R : \forall i, \cup_j v_{ij} = D$
    (every value appears in every row)
  - $C^C : \forall j, \cup_i v_{ij} = D$
    (every value appears in every column)
  - $C^B : \forall k, \cup (v_{ij} \mid ij \in B_k) = D$
    (every value appears in every block)
- Alternative representation: pairwise inequality constraints
  - $I^R : \forall i, j \neq j' : v_{ij} \neq v_{ij'}$
    (no value appears twice in any row)
  - $I^C : \forall j, i \neq i' : v_{ij} \neq v_{i'j}$
    (no value appears twice in any column)
  - $I^B : \forall k, ij \in B_k, i'j' \in B_k, ij \neq i'j' : v_{ij} \neq v_{i'j'}$
    (no value appears twice in any block)

Advantage of the second choice: all binary constraints!

Sudoku Constraint Network

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}



Solving Constraint Problems

Algorithms at last!

1. Systematic search
   - Generate and test
   - Backtracking
2. Constraint propagation (consistency)
3. Variable ordering heuristics
4. Value ordering heuristics
5. Backjumping and dependency-directed backtracking

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Generate and Test: Sudoku

- Try every possible assignment of domain elements to variables until you find one that works:

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	1

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	2

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	3

…

- Doesn't check constraints until all variables have been instantiated
- Very inefficient way to explore the space of possibilities (4^7 for this trivial Sudoku puzzle, mostly illegal)

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Systematic Search: Backtracking

(a.k.a. depth-first search!)

- Consider the **variables** in some order
  1. Pick an unassigned variable
  2. Give it a provisional value
  3. That it is consistent with all of the constraints
- If no such assignment can be made, we've reached a dead end and need to backtrack to the previous variable
- Continue this process until:
  - A solution is found, or
  - We backtrack to the initial variable and have exhausted all possible values

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Problems with Backtracking

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

- Thrashing: keep repeating same failed variable assignments
  - Consistency checking can help
  - Intelligent backtracking schemes can also help
- Inefficiency: can spend time exploring areas of search space that aren't likely to succeed
  - **Variable ordering can help**
  - IF there's a meaningful way to order them

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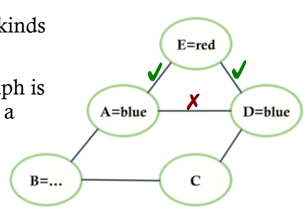
Consistency

- An assignment of values to variables is said to be **consistent** if no constraints are violated
- There are multiple kinds of consistency
- Once the whole graph is consistent, we have a solution



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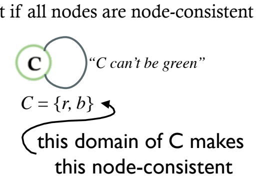
Node Consistency

- **Node consistency:** every value in **node X**'s domain (every value we think it might take) is consistent with *X's unary constraints*
  - A graph is node-consistent if all nodes are node-consistent
  - E.g., C can't be green
  - C = {red, ~~green~~, blue}



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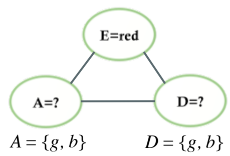
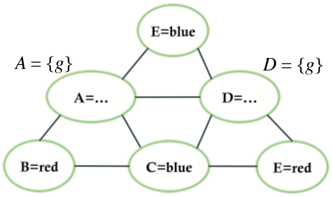
Arc Consistency

- **Arc consistency:**
- For every value *x* of X in Arc(X,Y):
  - **∃ *y*** for Y
  - That satisfies the constraint represented by the arc
  - A graph is arc-consistent if all arcs are arc-consistent



Arc Consistency: Example

- For every value *x* of X in Arc(X,Y): **∃ *y*** for Y that satisfies the constraint represented by the arc

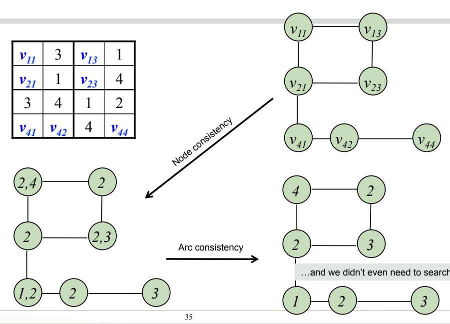


- Is this instantiation arc-consistent?

Constraint Propagation

- How do we find a set of consistent assignments?
- We perform **constraint propagation**
  - That is, we repeatedly reduce the domain of each variable to be consistent with its arcs
- Constraints reduce # of **legal values for a variable**
  - Which may then reduce legal values of another variable
    - Then another, then another…
- Key idea: **local consistency**
  - Enforce *nearby* constraints
  - Propagate

Constraint Propagation: Sudoku



Example: Map-Coloring

- Variables: WA, NT, Q, NSW, V, SA, T
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors
  - Ex: $WA \neq NT$

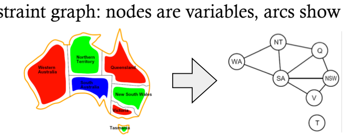
$(WA, NT) \in \{(red, green), (red, blue), (green, blue), (green, red), (blue, red)\}$

- Solutions are assignments satisfying all constraints, e.g.:

$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints



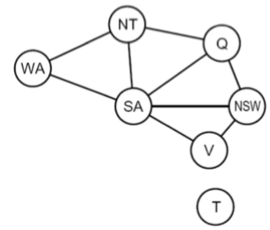
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

Standard Search Formulation

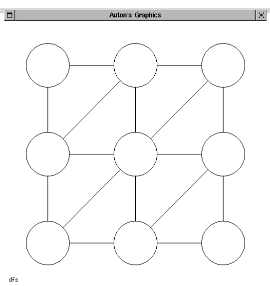
- Standard search formulation of CSPs (incremental)
- Let's start with a straightforward, dumb approach, then fix it
- **States** are defined by the **values assigned so far** (ex: WA=*red*, T=*red* is a state)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

Search Methods

- What does BFS do?
- What does DFS do?



DFS & BFS: not good!



Backtracking Search

- So how do we improve it?
- Idea 1: Only consider a single variable at each point
  - Variable assignments are commutative, so fix the ordering
    - Ex: [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there now?
- Idea 2: Only allow legal assignments at each point
  - Consider only values which do not conflict with existing assignments
  - Might have to do some computation to figure out whether a value is ok
  - "Incremental goal test"

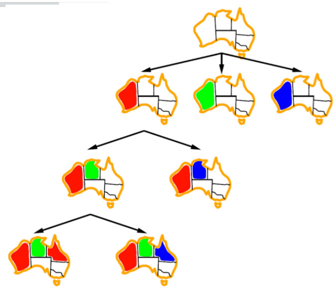
Backtracking Search

- Idea 1: Only consider a single variable at each point
- Idea 2: Only allow legal assignments at each point
- DFS for CSPs with these two improvements is called **backtracking search**
  - We *backtrack* when there's no legal assignment for the next variable
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

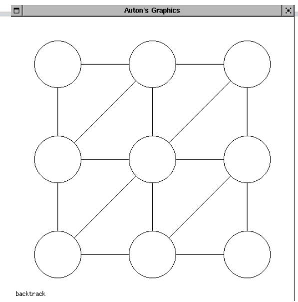
Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

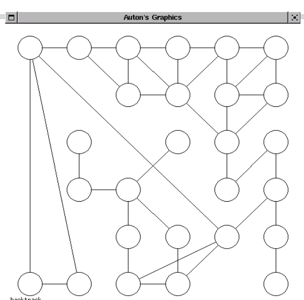
Backtracking Example



Backtracking



Good Enough?

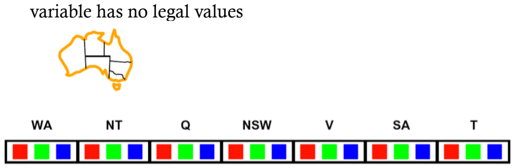


Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

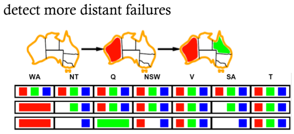
Forward Checking

- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints); terminate when any variable has no legal values

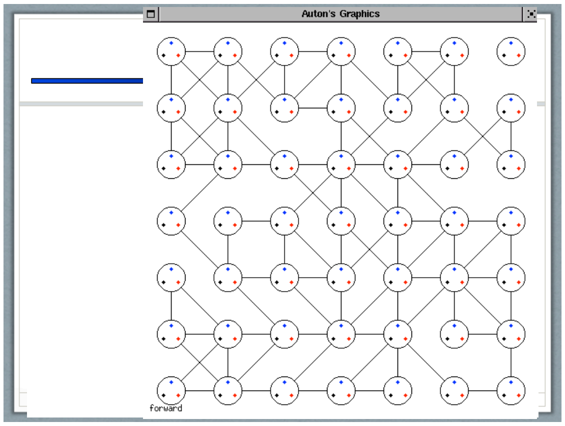


Forward Checking

- Propagates information from assigned to adjacent unassigned variables
- But doesn't detect more distant failures

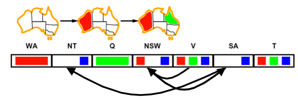


- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- Constraint propagation repeatedly enforces constraints **locally** – this is a local maximum!



Arc Consistency

- Simplest form of propagation makes each **arc** consistent
  - X → Y is *consistent* iff for every value x there is *some* allowed y



- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What's the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

K-consistency

- K-consistency generalizes the notion of arc consistency to sets of **more than two variables**
- A graph is **K-consistent** if, for legal values of any K-1 variables in the graph, and for any K^{th} variable V_k, there is a legal value for V_k
- **Strong** K-consistency = J-consistency for all J≤K
- Node consistency = strong 1-consistency
- Arc consistency = strong 2-consistency
- Path consistency = strong 3-consistency

Why Do We Care?

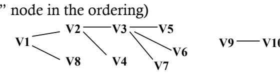
1. A strongly N-consistent CSP with N variables can be solved **without backtracking**
2. For any CSP that is strongly K-consistent:
   - If we find an appropriate variable ordering (one with "small enough" branching factor)
   - We can solve the CSP without backtracking

Ordered Constraint Graphs

- Select a variable ordering, V_1, …, V_n
- **Width of a node** in this OCG is the number of arcs leading to *earlier* variables:
  - $w(V_i) = \text{Count}((V_i, V_k) \mid k < i)$
- **Width of the OCG** is the maximum width of any node:
  - $w(G) = \text{Max}(w(V_i)),\ 1 \le i \le N$
- **Width of an unordered CG** is the minimum width of all orderings of that graph ("best you can do")

Tree-Structured Constraint Graph

- A **constraint tree** rooted at V_1 satisfies:
  - There exists an ordering V_1, …, V_n such that **every node has zero or one parents** (i.e., each node only has constraints with at most one "earlier" node in the ordering)

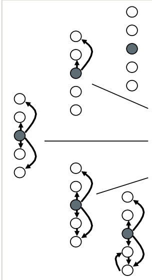


  - Also known as an *ordered constraint graph with width 1*
- If this constraint tree is also node- and arc-consistent (a.k.a. *strongly 2-consistent*), it can be solved without backtracking
  - (More generally, if the ordered graph is strongly k-consistent, and has width w < k, then it can be solved without backtracking.)

So What If We Don't Have a Tree?

- Answer #1: Try **interleaving** constraint propagation and backtracking
- Answer #2: Try using **variable-ordering** heuristics to improve search
- Answer #3: Try using **value-ordering** heuristics during variable instantiation
- Answer #4: See if **iterative repair** works better
- Answer #5: Try using **intelligent backtracking** methods

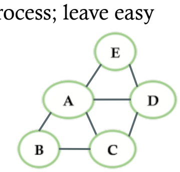
Variations on Interleaving Constraint Propagation and Search



Generate and Test	No constraint propagation: assign all variable values, **then** test constraints
Simple Backtracking	Check constraints only for variables "up the tree"
Forward Checking	Check constraints for *immediate* neighbors "down the tree"
Partial Lookahead	Propagate constraints forward "down the tree"
Full Lookahead	Ensure complete arc consistency after each instantiation (AC-3)

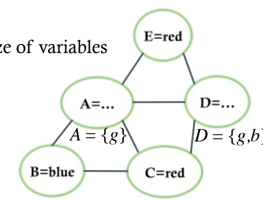
Possible Variable Orderings

- **Intuition**: choose variables that are highly constrained early in the search process; leave easy ones for later.
- How?
  - **Minimum width ordering** (MWO): identify OCG with minimum width
  - **Maximum cardinality ordering**: approximation of MWO that's cheaper to compute: order variables by decreasing cardinality (a.k.a. **degree heuristic**)



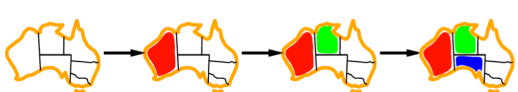
Possible Variable Orderings

- **Fail first principle** (FFP): choose variable with the fewest values (a.k.a. **minimum remaining values** (MRV))
  - **Static** FFP: use domain size of variables
  - **Dynamic** FFP (**search rearrangement method**): At each choice, select the variable with the fewest remaining values



Minimum Width

- Or "minimum remaining values" (MRV):
  - Choose the variable with the *fewest remaining* legal values



- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering

Variable Orderings II

- **Maximal stable set**: find largest set of variables with no constraints between them, save these for last
- **Cycle-cutset tree creation**: Find a set of variables that, once instantiated, leave a tree of uninstantiated variables; solve these, then solve the tree without backtracking
- **Tree decomposition**: Construct a tree-structured set of connected subproblems

Value Ordering

- **Intuition**: Choose **values** that are the least constrained early on, leaving the most legal values in later variables

1. **Maximal options method** (a.k.a. **least-constraining-value** heuristic): Choose the value that leaves the most legal values for not-yet-instantiated variables
2. **Min-conflicts**: For iterative repair search (Coming up)
3. Symmetry: Introduce **symmetry-breaking constraints** to constrain search space to 'useful' solutions (don't examine more than one symmetric/isomorphic solution)

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Iterative Repair

- Start with an initial complete (but probably invalid) assignment
- Repair locally
- **Min-conflicts:** Select new values that minimally conflict with the other variables
  - Use in conjunction with hill climbing or simulated annealing or…
- **Local maxima strategies**
  - Random restart
  - Random walk

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Min-Conflicts Heuristic

- Iterative repair method
  1. Find some "reasonably good" initial solution
     - E.g., in N-queens problem, use greedy search through rows, putting each queen where it conflicts with the smallest number of previously placed queens, breaking ties *randomly*
  2. Pick a variable in conflict (randomly)
  3. Select a new value that *minimizes* the number of constraint violations
     - O(N) time and space
  4. Repeat steps 2 and 3 until done

Min-Conflicts Heuristic

- Iterative repair method
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     - E.g., in N-queens problem, use greedy search through rows, putting each queen where it conflicts with the smallest number of previously placed queens, breaking ties *randomly*
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Performance depends on quality and informativeness of initial assignment; inversely related to distance to solution

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Intelligent Backtracking

- **Backjumping**: if V_j fails, jump back to the variable V_i with greatest i such that the constraint (V_i, V_j) fails (i.e., most recently instantiated variable in conflict with V_j)
- **Backchecking**: keep track of incompatible value assignments computed during backjumping
- **Backmarking**: keep track of which variables led to the incompatible variable assignments for improved backchecking

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Challenges

- What if not all constraints can be satisfied?
  - Hard vs. soft constraints
  - Degree of constraint satisfaction
  - Cost of violating constraints
- What if constraints are of different forms?
  - Symbolic constraints
  - Numerical constraints [*constraint solving*]
  - Temporal constraints
  - Mixed constraints

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More Challenges

- What if constraints are represented intensionally?
  - Cost of evaluating constraints (time, memory, resources)
- What if constraints/variables/values change over time?
  - Dynamic constraint networks
  - Temporal constraint networks
  - Constraint repair
- What if you have multiple agents or systems involved?
  - Distributed CSPs
  - Localization techniques

Questions?

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