Probabilistic Reasoning

AI Class 9 (Ch. 13)

Based on slides by Dr. Marie desJardin and Dr. Tim Oates. Some material also adapted from slides by Dr. Matuszek @ Villanova University, which are based in part on www.csc.calpoly.edu/~fkurfess/Courses/CSC-481/W02/Slides/Uncertainty.ppt and www.cs.umbc.edu/courses/graduate/671/fall05/slides/c18_prob.ppt

Today's Class

- Probability theory
- Probability notation
- Bayesian inference
  - From the joint distribution
  - Using independence / factoring
  - From sources of evidence

Probabilistic inference: finding *posterior probability* for a proposition, given observed evidence.

– R&N 490

Today's Class

We don't (can't!) know everything about most problems.

- Most problems are not:
  - Deterministic
  - Fully observable
- Or, we can't calculate everything.
  - Continuous problem spaces

Probability lets us understand. quantify, and work with this uncertainty.

Bayesian Reasoning

- Posteriors and priors
- What is inference?
- What is uncertainty?
- When/why use probabilistic reasoning?
- What is induction?
- What is the probability of two independent events?
- Frequentist/objectivist/subjectivist assumptions

Sources of Uncertainty

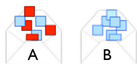
- Uncertain **inputs**
  - Missing data
  - Noisy data
- Uncertain **knowledge**
  - >1 cause → >1 effect
  - Incomplete knowledge of conditions or effects
  - Incomplete knowledge of causality
  - Probabilistic effects
- Uncertain **outputs**
  - Default reasoning (even deduction) is uncertain
  - Abduction & induction inherently uncertain
  - Incomplete deductive inference can be uncertain

Probabilistic reasoning only gives **probabilistic results** (summarizes uncertainty from various sources)

Decision Making with Uncertainty

- **Rational** behavior: for each possible action,
  - Identify possible outcomes
  - Compute **probability** of each outcome
  - Compute **utility** of each outcome
    - "goodness" or "desirability" per some formally specified definition
  - Compute probability-weighted **(expected) utility** of possible outcomes for each action
  - Select the action with the highest expected utility (principle of **Maximum Expected Utility**)

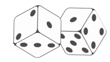
Also the definition of "rational" for deterministic decision-making!



Probability

- **World:** The complete set of possible states
- **Random variables:** Problem aspects that take a value
  - "The number of blue squares we are holding," *B*
  - "The combined value of two dice we rolled," *C*
- **Event:** Something that happens
- **Sample Space:** All the things (outcomes) that could happen in some set of circumstances
  - Pull 2 squares from envelope A: what is the sample space?
  - How about envelope B?
- **World, redux:** A *complete assignment of values to variables*

Basic Probability



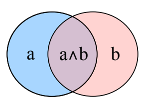
- Each P is a non-negative value in [0,1]
  - P({1,1}) = 1/36
- Total probability of the sample space is 1
  - P({1,1}) + P({1,2}) + P({1,3}) + … + P({6,6}) = 1
- For mutually exclusive events, the probability for at least one of them is the sum of their individual probabilities
  - P(sunny) ∨ P(cloudy) = P(sunny) + P(cloudy)
- Experimental probability: Based on frequency of past events
- Subjective probability: Based on expert assessment

commons.wikimedia.org/wiki/File:2-Dice-Icon.svg

Why Probabilities Anyway?

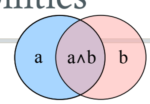
3 simple axioms → all rules of probability theory*

1. All probabilities are between 0 and 1.
   - $0 \le P(a) \le 1$
2. Valid propositions (tautologies) have probability 1, and unsatisfiable propositions have probability 0.
   - $P(true) = 1$
   - $P(false) = 0$
3. The probability of a disjunction is:
   - $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



*Kolmogorov – en.wikipedia.org/wiki/Andrey_Kolmogorov
De Finetti, Cox, and Carnap have also provided compelling arguments for these axioms

Compound Probabilities



- Describe **independent** events
  - Do not affect each other in any way
- **Joint** probability of two independent events A and B
  $P(A \cap B) = P(A) * P(B)$ ← What do these say?
- **Union** probability of two independent events A and B
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  $= P(A) + P(B) - (P(A) * P(B))$

Pull two squares from envelope A. What is the probability that they are BOTH red?

Probability Theory

- **Random variables:**
  - Domain: possible values
- **Atomic event:**
  - Complete specification of a state
- **Prior probability:**
  - Degree of belief without any *new* evidence
- **Joint probability:**
  - Matrix of **combined probabilities of a set of variables,** P(*A*|*B*)

- Alarm (*A*), Burglary (*B*), Earthquake (*E*)
  - Boolean, discrete, continuous
- A=*true* ∧ B=*true* ∧ E=*false*:
  - alarm ∧ burglary ∧ ¬earthquake
- P(*B*) = 0.1
- P(*A*, *B*) =

	alarm	**¬alarm**
burglary	0.09	0.01
¬burglary	0.1	0.8

Probability Distributions

- A distribution is the probabilities of **all possible values** of a random variable
- Ex: weather can be sunny, rainy, cloudy, or snowy
  - P(Weather = sun) = 0.6
  - P(Weather = rain) = 0.1
  - P(Weather = cloud) = 0.29
  - P(Weather = snow) = 0.01
  - **P**(Weather) = <0.6, 0.1, 0.29, 0.01> ← shortcut
- **P**(Weather): **probability distribution on Weather**

Probability Theory: Definitions

- **Conditional probability:** Probability of some effect given that we know cause(s)
  - Example: $P(alarm|burglary)$
    - (Technically, we only know b is true, not causal, but…)
- Computing it:
  - $P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$
- $P(b)$: **normalizing constant**
  - (Later we'll call this alpha)

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Probability Theory: Definitions

- **Product rule**:
  - $P(a \wedge b) = P(a \mid b)\ P(b)$
- **Marginalizing** (summing out):
  - Finding distribution over *one* or a *subset* of variables
  - Marginal probability of B summed over all alarm states:
  - $P(B) = \Sigma_a P(B,\ a)$
- **Conditioning** over a subset of variables:
  - $P(B) = \Sigma_a P(B \mid a)\ P(a)$

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Try It…

	alarm	¬alarm
burglary	0.09	0.01
¬burglary	0.1	0.8

- **Cond'l probability**
  - P(effect, cause[s])
  - $P(a \mid b) = P(a \wedge b) / P(b)$
  - $P(b)$: **normalizing** constant ($1/\alpha$)
- **Product rule**:
  - $P(a \wedge b) = P(a \mid b)\ P(b)$
- **Marginalizing**:
  - $P(B) = \Sigma_a P(B, a)$
  - $P(B) = \Sigma_a P(B \mid a)\ P(a)$ (**conditioning**)

- $P(A \mid B)$ = ?
- $P(B \mid A)$ = ?
- $P(B \wedge A)$ = ?
- $P(A)$ = ?

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Example: Inference from the Joint

- $P(B \mid A) = \alpha\ P(B, A)$
  $= \alpha\ [P(B, A, E) + P(B, A, \neg E)]$
  $= \alpha\ [(.01,\ .01) + (.08,\ .09)]$
  $= \alpha\ [(.09,\ .1)]$

	A		¬A	
	E	**¬E**	**E**	**¬E**
B	0.01	0.08	0.001	0.009
¬B	0.01	0.09	0.01	0.79

- Since
  $P(B \mid A) + P(\neg B \mid A) = 1$, $\alpha = 1 / (0.09 + 0.1) = 5.26$
  (i.e., $P(A) = 1/\alpha = 0.19$)
- $P(B \mid A) = 0.09 * 5.26 = 0.474$
- $P(\neg B \mid A) = 0.1 * 5.26 = 0.526$

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Exercise: Inference from the Joint

- Queries: what is…
  - The prior probability (knowing noth…
  - The prior probability of *study*?
  - The conditional probability of *prepare*… …ven *study* and *smart*?

Where do these come from?

***P*(smart ∧ study ∧ prep)**	***smart***		***¬smart***	
	study	***¬study***	***study***	***¬study***
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

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Exercise: Inference from the joint

***P*(smart ∧ study ∧ prep)**	***smart***		***¬smart***	
	study	***¬study***	***study***	***¬study***
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Queries:

- **What is the prior probability of *smart*?**
- What is the prior probability of *study*?
- What is the conditional probability of *prepared*, given *study* and *smart*?

P(smart) = .432 + .16 + .048 + .16 = **0.8**

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Exercise: Inference from the joint

$P(smart \wedge study \wedge prep)$	*smart*		*¬smart*	
	study	*¬study*	*study*	*¬study*
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Queries:

- What is the prior probability of *smart*?
- **What is the prior probability of *study*?**
- What is the conditional probability of *prepared*, given *study* and *smart*?

P(study) = .432 + .048 + .084 + .036 = **0.6**

Exercise: Inference from the joint

$P(smart \wedge study \wedge prep)$	*smart*		*¬smart*	
	study	*¬study*	*study*	*¬study*
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Queries:

- **What is the conditional probability of *prepared*, given *study* and *smart*?**

P(prep | smart,study) = P(prep, smart, study)/P(smart, study)
= .432 / (.432 + .048)
= **0.9**

Independence: ⫫

- **Independent:** Two sets of propositions that do not affect each others' probabilities
- Easy to calculate **joint** and **conditional** probability of independence:
  $(A, B) \Leftrightarrow P(A \wedge B) = P(A)\,P(B)$ or $P(A \mid B) = P(A)$
- Examples:
  - A = alarm, B = burglary, E = earthquake
  - M = moon phase, L = light level

$A ⫫ B ⫫ E = f$
$M ⫫ L = f$
$A ⫫ M = t$

Independence Example

- ***{moon-phase, light-level}* ⫫ *{burglary, alarm, earthquake}***
  - But maybe burglaries increase in low light
  - But, *if* we know the light level, *moon-phase* ⫫ *burglary*
  - *Once* we're burglarized, light level doesn't affect whether the alarm goes off; {*light-level*} ⫫ {*alarm*}
- We need:
  1. A more complex notion of independence
  2. Methods for reasoning about these kinds of (common) relationships

Exercise: Independence

$P(smart \wedge study \wedge prep)$	*smart*		*¬smart*	
	study	*¬study*	*study*	*¬study*
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- Queries:
  - Is *smart* independent of
    - P(*smart* | *study*) = P(*smart*)
  - Is *prepared* independent
    - P(*prep* | *study*) = P(*prep*)

Smart	Study		
t	*t*	0.432 + 0.48	0.480
t	*f*	0.16 + 0.16	0.32
f	*t*	0.084 + 0.008	0.092
f	*f*	0.036 + 0.72	0.756

Conditional Probabilities

- Describes **dependent** events
  - Affect each other in some way
- Typical in the real world
- If we know some event has occurred, what does that tell us about the likelihood of another event?

Conditional Independence

- *moon-phase* and *burglary* are **conditionally independent given** *light-level*
  - That is, $M \perp\!\!\!\perp B$ if we already know L
- Conditional independence is:
  - Weaker than absolute independence
  - Useful in decomposing full joint probability distributions

Conditional Independence

- **Absolute** independence: A ⫫ B, if:
  - $P(A \wedge B) = P(A)\, P(B)$
  - Equivalently, $P(A) = P(A \mid B)$ and $P(B) = P(B \mid A)$
- A and B are **conditionally independent** given C if:
  - $P(A \wedge B \mid C) = P(A \mid C)\, P(B \mid C)$
- This lets us decompose the joint distribution:
  - $P(A \wedge B \wedge C) = P(A \mid C)\, P(B \mid C)\, P(C)$
- What does this mean?

Exercise: Conditional Independence

***P*(*smart* ∧ *study* ∧ *prep*)**	***smart***		***¬smart***	
	study	***¬study***	***study***	***¬study***
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- Queries:
  - Is *smart* conditionally independent of *prepared*, given *study*?
  - Is *study* conditionally independent of *prepared*, given *smart*?

Bayes' Rule

- Derive the probability of an **event** given **another event**
  - Assumption of attribute independency (Naïve assumption): Naïve Bayes assumes that all *attributes* are independent.
- Bayes' rule is derived from the product rule: R&N 495
  - $P(Y \mid X) = P(X \mid Y)\, P(Y) / P(X)$
- Often useful for **diagnosis**. If we have:
  - X = (observed) effects, Y = (hidden) causes
  - A model for how causes lead to effects: $P(X \mid Y)$
  - Prior beliefs about frequency of occurrence of effects: $P(Y)$
- We can reason abductively from effects to causes:
  - $P(Y \mid X)$

Naïve Bayes Algorithm

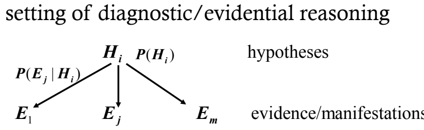
- Estimate the probability of each class:
  - Compute the posterior probability (Bayes rule)

$$P(c_i \mid D) = \frac{P(c_i)P(D \mid c_i)}{P(D)}$$

  - Choose the class with the highest probability
- Assumption of attribute independency (Naïve assumption): Naïve Bayes assumes that all of the attributes are independent.

Bayesian Inference

- In the setting of diagnostic/evidential reasoning



  - Know: prior probability of hypothesis $P(H_i)$
    conditional probability $P(E_j \mid H_i)$
  - Want to compute the *posterior probability* $P(H_i \mid E_j)$
- Bayes' theorem (formula 1):

$$P(H_i \mid E_j) = P(H_i)P(E_j \mid H_i) / P(E_j)$$

Simple Bayesian Diagnostic Reasoning

- Knowledge base:
  - Evidence / manifestations: $E_1, \ldots E_m$
  - Hypotheses / disorders: $H_1, \ldots H_n$
    - E_j and H_i are **binary**; hypotheses are **mutually exclusive** (non-overlapping) and **exhaustive** (cover all possible cases)
  - Conditional probabilities: $P(E_j \mid H_i)$, $i = 1, \ldots n$; $j = 1, \ldots m$
- Cases (evidence for a particular instance): $E_1, \ldots, E_m$
- Goal: Find the hypothesis H_i with the highest posterior
  - $\text{Max}_i\, P(H_i \mid E_1, \ldots, E_m)$

Priors

- Four values total here:
  - $P(H|E) = (P(E|H) * P(H)) / P(E)$
- $P(H|E)$ — what we want to compute
- Three we already know, called the *priors*
  - $P(E|H)$
  - $P(H)$
  - $P(E)$

(In ML we use the training set to estimate the priors)

Bayesian Diagnostic Reasoning II

- Bayes' rule says that
  - $P(H_i \mid E_1, \ldots, E_m) = P(E_1, \ldots, E_m \mid H_i)\, P(H_i) / P(E_1, \ldots, E_m)$
- Assume each piece of evidence E_i is conditionally independent of the others, **given** a hypothesis H_i, then:
  - $P(E_1, \ldots, E_m \mid H_i) = \prod^{l}_{j=1} P(E_j \mid H_i)$
- If we only care about relative probabilities for the H_i, then we have:
  - $P(H_i \mid E_1, \ldots, E_m) = \alpha\, P(H_i) \prod^{l}_{j=1} P(E_j \mid H_i)$

Bayes Example: Diagnosing Meningitis

$$P(H_i \mid E_j) = P(H_i)P(E_j \mid H_i) / P(E_j)$$

- Your patient comes in with a stiff neck.
- Is it meningitis?
- Suppose we know that
  - Stiff neck is a symptom in 50% of meningitis cases
  - Meningitis (m) occurs in 1/50,000 patients
  - Stiff neck (s) occurs in 1/20 patients
- So probably not. But specifically?

Bayes Example: Diagnosing Meningitis

$$P(H_i \mid E_j) = P(H_i)P(E_j \mid H_i) / P(E_j)$$

- Stiff neck is a symptom in 50% of meningitis cases
- Meningitis (m) occurs in 1/50,000 patients
- Stiff neck (s) occurs in 1/20 patients

- Then
  - $P(s|m) = 0.5$, $P(m) = 1/50000$, $P(s) = 1/20$
  - $P(m|s) = (P(s|m)\, P(m))/P(s)$
    $= (0.5 \times 1/50000) / 1/20 = .0002$
- So we expect that one in 5000 patients with a stiff neck to have meningitis.

Analysis of Naïve Bayes Algorithm

- Advantages:
  - Sound theoretical basis
  - Works well on numeric and textual data
  - Easy implementation and computation
  - Has been effective in practice (e.g., typical spam filter)

Limitations of Simple Bayesian Inference

- Cannot easily handle multi-fault situations, nor cases where intermediate (hidden) causes exist:
  - Disease D causes syndrome S, which causes correlated manifestations M_1 and M_2
- Consider a composite hypothesis $H_1 \wedge H_2$, where H_1 and H_2 are independent. What is the relative posterior?
  - $P(H_1 \wedge H_2 \mid E_1, \ldots, E_m) = \alpha \, P(E_1, \ldots, E_m \mid H_1 \wedge H_2) \, P(H_1 \wedge H_2)$
    $= \alpha \, P(E_1, \ldots, E_m \mid H_1 \wedge H_2) \, P(H_1) \, P(H_2)$
    $= \alpha \prod^{l}_{m=1} P(E_m \mid H_1 \wedge H_2) \, P(H_1) \, P(H_2)$
- How do we compute $P(E_j \mid H_1 \wedge H_2)$??

Limitations of Simple Bayesian Inference II

- Assume H1 and H2 are independent, given E1, …, Ej?
  - $P(H_1 \wedge H_2 \mid E_1, \ldots, E_j) = P(H_1 \mid E_1, \ldots, E_j) \, P(H_2 \mid E_1, \ldots, E_j)$
- This is a very unreasonable assumption
  - Earthquake and Burglar are independent, but *not* given Alarm:
    - P(burglar | alarm, earthquake) << P(burglar | alarm)
- Simple application of Bayes' rule doesn't handle causal chaining:
  - A: this year's weather; B: cotton production; C: next year's cotton price
  - A influences C indirectly: A→ B → C
  - P(C | B, A) = P(C | B)
- Need a richer representation to model interacting hypotheses, conditional independence, and causal chaining
- Next time: conditional independence and Bayesian networks!