Constraint Satisfaction

AI Class 7 — Ch. 6.1–6.4 (skip 6.3.3))

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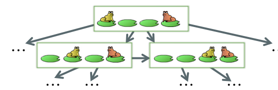
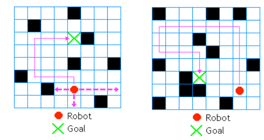
Bookkeeping

- HW 2 out: Search
  - Due 10/3, 11:59pm
- Jumping Frogs
- Ricochet Robots





Today's Class

- Constraint Satisfaction Problems
  - A.K.A., Constraint Processing / CSP paradigm
- Algorithms for CSPs
- Search Terminology

> **Constraint** (n): A relation … between the values of one or more mathematical variables (e.g., x>3 is a constraint on x). **Constraint satisfaction** assigns values to variables so that all constraints are true.
>
> *– http://foldoc.org/constraint*

Constraint Satisfaction

- Con•straint /kənˈstrānt/, (noun):
  - Something that limits or restricts someone or something.[1]
  - Control that limits or restricts someone's actions or behavior.[1]
  - A relation … between the values of one or more mathematical variables (e.g., x>3 is a constraint on x).[2]
  - Assigns values to variables so that all constraints are true.[2]
- General Idea
  - View a problem as a **set of variables**
  - To which we have to assign **values**
  - That satisfy a number of (problem-specific) **constraints**

[1] Merriam-Webster online.
[2] The Free Online Computing Dictionary.

Overview

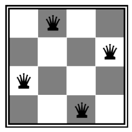
- **Constraint satisfaction**: a problem-solving paradigm
- Constraint programming, constraint satisfaction problems (CSPs), constraint logic programming…
- Algorithms for CSPs
  - Backtracking (systematic search)
  - Constraint propagation (k-consistency)
  - Variable and value ordering heuristics
  - **Backjumping and dependency-directed backtracking**

Search Vocabulary

- We've talked about caring about goals vs. paths
- These correspond to…
  - **Planning**: sequences of actions
    - The path to the goal is the important thing
    - Paths have various costs, depths
    - Heuristics to guide, fringe to keep backups
  - **Identification**: assignments to variables representing unknowns
    - The goal itself is important, not the path
- CSPs are specialized for identification problems

Slightly Less Informal Definition of CSP

- **CSP** = Constraint Satisfaction Problem
- Given:
  1. A finite set of **variables**
  2. Each with a **domain** of possible value (often finite)
  3. A set of **constraints** that limit the values the variables can take on
- **Solution**: an assignment of a value to each variable such that the constraints are all satisfied.

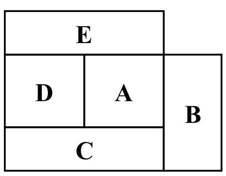


CSP Applications

- Decide if a solution exists
- Find some solution
- Find all solutions
- Find the "best solution"
  - According to some metric (objective function)
  - Does that mean "optimal"?

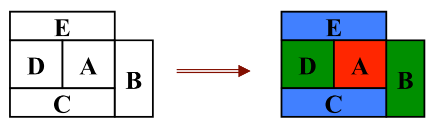
Informal Example: Map Coloring

- Color a map, such that:
  - Using three colors (red, green, blue)
  - No two adjacent regions have the same color



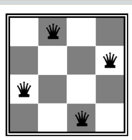
Map Coloring II

- Variables: A, B, C, D, E
- Domains: RGB = {red, green, blue}
- Constraints: A≠B, A≠C, A ≠ E, A ≠ D, B ≠ C, C ≠ D, D ≠ E
- One solution: A=red, B=green, C=blue, D=green, E=blue



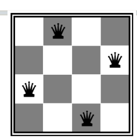
Slightly Less Informal

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - **State** is defined by variables X_i with values from a domain D
    - Sometimes D depends on i
  - Goal test is a **set of constraints** specifying allowable combinations of values for subsets of variables



Example: N-Queens (1)

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints:

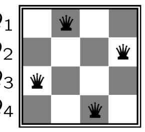


$$\forall i, j, k \ \ (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens (2)

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, ... N\}$
  - Constraints:

Q_1
Q_2
Q_3
Q_4



Implicit: $\forall i, j$ $\text{non-threatening}(Q_i, Q_j)$

-or-

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$

...

Example: SATisfiability

- Given a set of propositions containing variables, find an assignment of the variables to {false, true} that satisfies them.
- For example, the clauses:
  - (A ∨ B ∨ ¬C) ∧ (¬A ∨ D)
  - (equivalent to (C → A) ∨ (B ∧ D → A)

are satisfied by

A = false

B = true

C = false

D = false

Real-World Problems

- Scheduling
- Temporal reasoning
- Building design
- Planning
- Optimization/ satisfaction
- Vision
- Graph layout
- Network management
- Natural language processing
- Molecular biology / genomics
- VLSI design

Formal Definition: Constraint Network (CN)

A **constraint network** (CN) consists of

- A set of variables $X = \{x_1, x_2, \ldots x_n\}$
  - Each with an associated domain of values $\{d_1, d_2, \ldots d_n\}$.
  - The domains are typically finite
- A set of constraints $\{c_1, c_2 \ldots c_m\}$ where
  - Each constraint defines a **predicate which is a relation** over a particular subset of X.
  - E.g., c_i involves variables $\{X_{i1}, X_{i2}, \ldots X_{ik}\}$ and defines the relation $R_i \subseteq D_{i1} \times D_{i2} \times \ldots D_{ik}$
- **Unary** constraint: only involves one variable
- **Binary** constraint: only involves two variables

Formal Definition of a CN (cont.)

- Instantiations
  - An **instantiation** of a subset of variables S is an assignment of a value in its domain to each variable in S
  - An instantiation is **legal** iff it does not violate any constraints
- A **solution** is an instantiation of all of the variables in the network

Typical Tasks for CSP

- Solutions:
  - Does a solution exist?
  - Find one solution
  - Find all solutions
  - Given a partial instantiation, do any of the above
- Transform the CN into an equivalent CN that is easier to solve.

Binary CSP

- **Binary CSP**: all constraints are binary or unary
- Can convert a non-binary CSP → binary CSP by:
  - Introducing additional variables
  - Dual graph construction: one variable for each constraint; one binary constraint for each pair of original constraints that share variables
- Can represent a binary CSP as a **constraint graph** with:
  - A node for each variable
  - An arc between two nodes iff there is a constraint involving the two variables
  - Unary constraint appears as a self-referential arc

Example: Sudoku

	3		1
	1		4
3	4	1	2
		4	

Running Example: Sudoku

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

- Variables and their domains
  - v_{ij} is the value in the jth cell of the ith row
  - $D_{ij} = D = \{1, 2, 3, 4\}$
- Blocks:
  - $B_1 = \{11, 12, 21, 22\}, \ldots, B_4 = \{33, 34, 43, 44\}$
- Constraints (implicit/intensional)
  - $C^R : \forall i, \cup_j v_{ij} = D$ (every value appears in every row)
  - $C^C : \forall j, \cup_j v_{ij} = D$ (every value appears in every column)
  - $C^B : \forall k, \cup (v_{ij} \mid ij \in B_k) = D$ (every value appears in every block)
- Alternative representation: pairwise inequality constraints
  - $I^R : \forall i, j \neq j' : v_{ij} \neq v_{ij'}$ (no value appears twice in any row)
  - $I^C : \forall j, i \neq i' : v_{ij} \neq v_{i'j}$ (no value appears twice in any column)
  - $I^B : \forall k, ij \in B_k, i'j' \in B_k, ij \neq i'j' : v_{ij} \neq v_{i'j'}$ (no value appears twice in any block)
- Advantage of the second representation: all binary constraints!

Sudoku Constraint Network

	3		1
	1		4
3	4	1	2
		4	

Sudoku Constraint Network

v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
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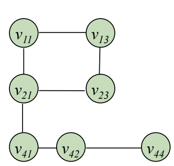


Solving Constraint Problems

- Systematic search
  - Generate and test
  - Backtracking
- Constraint propagation (consistency)
- Variable ordering heuristics
- Value ordering heuristics
- Backjumping and dependency-directed backtracking

Generate and Test: Sudoku

- Try each possible combination until you find one that works:

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	1

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	2

1	3	1	1
1	1	1	4
3	4	1	2
1	1	4	3

…

- Doesn't check constraints until all variables have been instantiated
- Very inefficient way to explore the space of possibilities (4^7 for this trivial Sudoku puzzle, most illegal)

Systematic Search: Backtracking

(a.k.a. depth-first search!)

- Consider the variables in some order
- Pick an unassigned variable and give it a provisional value such that it is consistent with all of the constraints
- If no such assignment can be made, we've reached a dead end and need to backtrack to the previous variable
- Continue this process until:
  - A solution is found, or
  - We backtrack to the initial variable and have exhausted all possible values

Problems with Backtracking

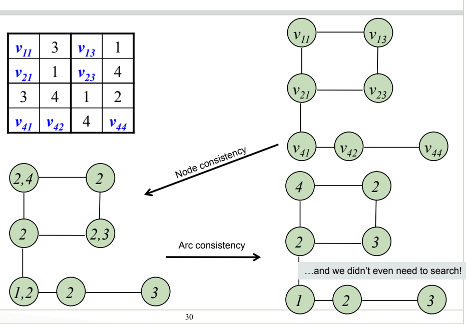
v_{11}	3	v_{13}	1
v_{21}	1	v_{23}	4
3	4	1	2
v_{41}	v_{42}	4	v_{44}

- Thrashing: keep repeating same failed variable assignments
  - Consistency checking can help
  - Intelligent backtracking schemes can also help
- Inefficiency: can spend time exploring areas of search space that aren't likely to succeed
  - Variable ordering can help
  - IF there's a meaningful way to order them

Consistency

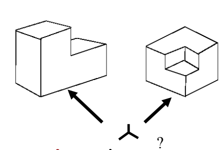
- Node consistency
  - Node X is **node-consistent** if every value in the X's domain is consistent with X's unary constraints
  - A graph is node-consistent if all nodes are node-consistent
- Arc consistency
  - Arc (X, Y) is **arc-consistent** if, for every value *x* of X, there is a value *y* for Y that satisfies the constraint represented by the arc.
  - A graph is arc-consistent if all arcs are arc-consistent.
- To create arc consistency, we perform **constraint propagation**: that is, we repeatedly reduce the domain of each variable to be consistent with its arcs

Constraint Propagation: Sudoku



Example: The Waltz Algorithm

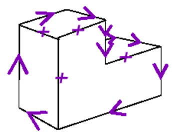
- Waltz algorithm is for interpreting line drawings of solid polyhedra
- Look at all intersections



- **Adjacent intersections** impose constraints on each other

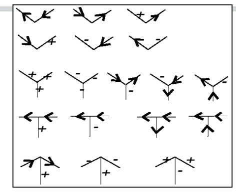
Waltz on Simple Scenes

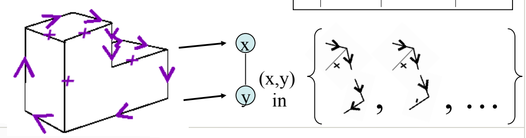
- Assume all objects:
  - Have no shadows or cracks
  - Three-faced vertices
  - "General position": no junctions change with small movements of the eye.
- Then each line on image is:
  - Boundary line (edge of an object) (→) with right hand of arrow denoting "solid" and left hand denoting "space" or
  - Interior convex edge (+)
  - Interior concave edge (-)



Legal Junctions

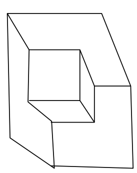
- Only certain junctions are physically possible
- How can we formulate a CSP to label an image?
- **Variables**: vertices
- **Domains**: junction labels
- **Constraints**: both ends of a line should have the same label

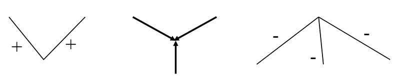




Line Drawings

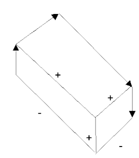
- Here are some illegal labelings for this shape:



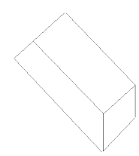


Line Drawings

- Propagate constraints repeatedly until a solution is found



A solution for one labeling problem



A labeling problem with no solution

Example: Map-Coloring

- Variables: WA, NT, Q, NSW, V, SA, T
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors

  $WA \neq NT$

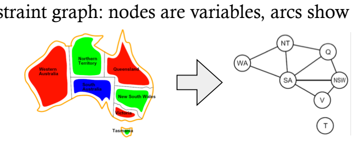
  $(WA, NT) \in \{(red, green), (red, blue), (green, red), \ldots\}$

- Solutions are assignments satisfying all constraints, e.g.:

  $\{WA = red, NT = green, Q = red,$
  $NSW = green, V = red, SA = blue, T = green\}$

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints



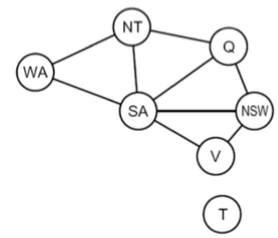
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

Standard Search Formulation

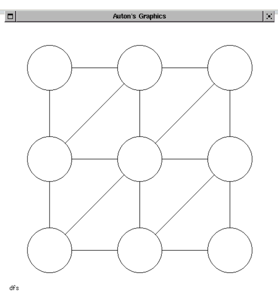
- Standard search formulation of CSPs (incremental)
- Let's start with a straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

Search Methods

- What does BFS do?
- What does DFS do?



DFS & BFS: not good!



Backtracking Search

- Idea 1: Only consider a single variable at each point
  - Variable assignments are commutative, so fix ordering
    - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there now?
- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok
  - "Incremental goal test"

Backtracking Search

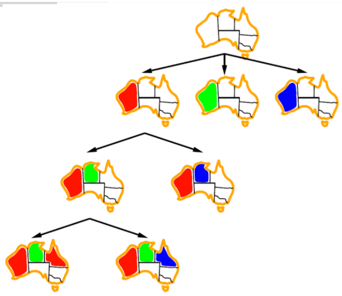
- Idea 1: Only consider a single variable at each point
- Idea 2: Only allow legal assignments at each point
- Depth-first search for CSPs with these two improvements is called *backtracking search*
  - Backtrack when there's no legal assignment for the next variable
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

Backtracking Search

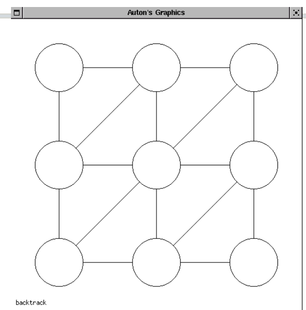
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment given CONSTRAINTS[csp] then
      add {var = value} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove {var = value} from assignment
  return failure
```

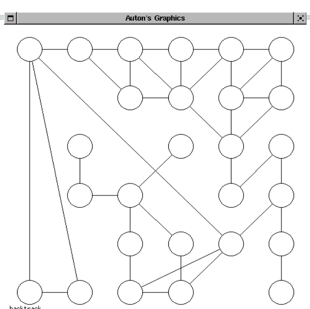
Backtracking Example



Backtracking



Good Enough?

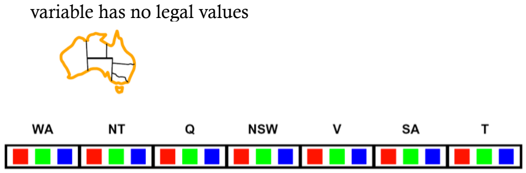


Improving Backtracking

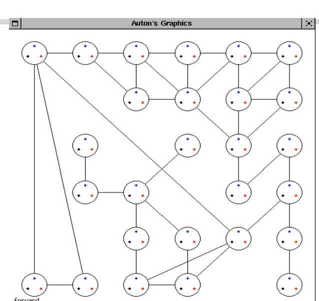
- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

Forward Checking

- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints); terminate when any variable has no legal values



Forward Checking



K-consistency

- K-consistency generalizes the notion of arc consistency to sets of more than two variables
  - A graph is K-consistent if, for legal values of any K-1 variables in the graph, and for any K^{th} variable V_k, there is a legal value for V_k
- **Strong** K-consistency = J-consistency for all J<=K
- Node consistency = strong 1-consistency
- Arc consistency = strong 2-consistency
- Path consistency = strong 3-consistency

Why Do We Care?

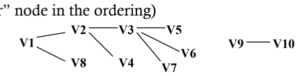
1. A strongly N-consistent CSP with N variables can be solved **without backtracking**
2. For any CSP that is strongly K-consistent:
   - If we find an appropriate variable ordering (one with "small enough" branching factor)
   - We can solve the CSP without backtracking

Ordered Constraint Graphs

- Select a variable ordering, $V_1, \ldots, V_n$
- **Width of a node** in this OCG is the number of arcs leading to earlier variables:
  - $w(V_i) = \text{Count}\ (\ (V_i, V_k)\ |\ k < i)$
- **Width of the OCG** is the maximum width of any node:
  - $w(G) = \text{Max}\ (w\ (V_i)),\ 1 <= i <= N$
- **Width of an unordered CG** is the minimum width of all orderings of that graph ("best you can do")

Tree-Structured Constraint Graph

- A **constraint tree** rooted at V_1 satisfies:
  - There exists an ordering $V_1, \ldots, V_n$ such that **every node has zero or one parents** (i.e., each node only has constraints with at most one "earlier" node in the ordering)



  - Also known as an *ordered constraint graph with width 1*
- If this constraint tree is also node- and arc-consistent (a.k.a. *strongly 2-consistent*), it can be solved without backtracking
  - (More generally, if the ordered graph is strongly k-consistent, and has width w < k, then it can be solved without backtracking.)

Proof Sketch for Constraint Trees

- Perform backtracking search in the order that satisfies the constraint tree condition
- Every node, when instantiated, is constrained only by at most one previous node
- Arc consistency tells us that there must be at least one legal instantiation in this case
  - (If there are no legal solutions, the arc consistency procedure will collapse the graph – some node will have no legal instantiations)
- Keep doing this for all *n* nodes, and you have a legal solution – without backtracking!

Variable Ordering

- **Intuition**: choose variables that are highly constrained early in the search process; leave easy ones for later
- **Minimum width ordering** (MWO): identify OCG with minimum width
- **Maximum cardinality ordering**: approximation of MWO that's cheaper to compute: order variables by decreasing cardinality (a.k.a. **degree heuristic**)
- **Fail first principle** (FFP): choose variable with the fewest values (a.k.a. **minimum remaining values** (MRV))
  - **Static** FFP: use domain size of variables
  - **Dynamic** FFP (**search rearrangement method**): At each point in the search, select the variable with the fewest remaining values

Minimum Width

- Or “minimum remaining values” (MRV):
  - Choose the variable with the fewest legal values

- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering

Variable Ordering II

- **Maximal stable set**: find largest set of variables with no constraints between them and save these for last
- **Cycle-cutset tree creation**: Find a set of variables that, once instantiated, leave a tree of uninstantiated variables; solve these, then solve the tree without backtracking
- **Tree decomposition**: Construct a tree-structured set of connected subproblems

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Value Ordering

- **Intuition**: Choose values that are the least constrained early on, leaving the most legal values in later variables
- **Maximal options method** (a.k.a. **least-constraining-value** heuristic): Choose the value that leaves the most legal values in uninstantiated variables
- **Min-conflicts**: For iterative repair search (Coming up)
- Symmetry: Introduce **symmetry-breaking constraints** to constrain search space to ‘useful’ solutions (don’t examine more than one symmetric/isomorphic solution)

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Iterative Repair

- Start with an initial complete (but invalid) assignment
- Hill climbing, simulated annealing
- Min-conflicts: Select new values that minimally conflict with the other variables
  - Use in conjunction with hill climbing or simulated annealing or…
- Local maxima strategies
  - Random restart
  - Random walk
  - Tabu search: don’t try recently attempted values

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Min-Conflicts Heuristic

- Iterative repair method
  1. Find some “reasonably good” initial solution
     – E.g., in N-queens problem, use greedy search through rows, putting each queen where it conflicts with the smallest number of previously placed queens, breaking ties *randomly*
  2. Find a variable in
  3. Select a new valu constraint violatio
     – O(N) time and spa
  4. Repeat steps 2 and 3 until done

Performance depends on quality and informativeness of initial assignment; inversely related to distance to solution

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Challenges

- What if not all constraints can be satisfied?
  - Hard vs. soft constraints
  - Degree of constraint satisfaction
  - Cost of violating constraints
- What if constraints are of different forms?
  - Symbolic constraints
  - Numerical constraints [constraint solving]
  - Temporal constraints
  - Mixed constraints

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More Challenges

- What if constraints are represented intensionally?
  - Cost of evaluating constraints (time, memory, resources)
- What if constraints/variables/values change over time?
  - Dynamic constraint networks
  - Temporal constraint networks
  - Constraint repair
- What if you have multiple agents or systems involved in constraint satisfaction?
  - Distributed CSPs
  - Localization techniques

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