Decision Making under Uncertainty
AI Class 10 (Ch. 15.1-15.2.1, 16.1-16.3)

Bookkeeping

- HW 3 out
  - Group work for non-programming parts!
  - Heavy on CSPs and probability
  - Forms groups today or in Piazza
- Soon: form project teams!
Today’s Class

• Making Decisions Under Uncertainty
  • Tracking Uncertainty over Time
  • Decision Making under Uncertainty

• Project groups, part 1 Left?
Sources of Uncertainty

- Uncertain inputs
  - Missing data
  - Noisy data
- Uncertain knowledge
  - >1 cause → >1 effect
  - Incomplete knowledge of causality
  - Probabilistic effects
- Uncertain outputs
  - Default reasoning (even deduction) is uncertain
  - Abduction & induction inherently uncertain
  - Incomplete deductive inference can be uncertain
  - Derived result is formally correct, but wrong in real world

Probabilistic reasoning only gives probabilistic results
(summarizes uncertainty from various sources)

Reasoning Under Uncertainty

- People make successful decisions all the time anyhow.
  - How?
  - More formally: how do we do reasoning under uncertainty, with inexact knowledge?
- Step one: understanding what we know
States and Observations

- We don't have a continuous view of the world
  - People don't either!

- We see things as a series of snapshots

- **Observations**, associated with **time slices**
  - $t_1, t_2, t_3, \ldots$

- Each snapshot contains all variables, observed or not
  - $X_i =$ (unobserved) state variables at time $t$; observation at $t$ is $E_i$

- This is **world state at time** $t$
Temporal Probabilistic Agent

Time and Uncertainty

- The world changes
  - Examples: diabetes management, traffic monitoring
- Tasks: **track** it; **predict** it
- Basic idea:
  - Copy state and evidence variables for each time step
  - Model uncertainty in change over time
  - Incorporate new observations as they arrive
Time and Uncertainty

• Basic idea:
  • Copy state and evidence variables for each time step
  • Model uncertainty in change over time
  • Incorporate new observations as they arrive

• $X_t =$ unobservable state variables at time $t$:
  BloodSugar, StomachContents,

• $E_t =$ evidence variables at time $t$:
  MeasuredBloodSugar, PulseRate, FoodEaten,

• Assuming discrete time steps

States, Slightly More formally

• Process of change is viewed as series of snapshots
  • Time slices
  • Each describing the state of the world at a particular time

• Each time slice is represented by a set of random variables indexed by $t$:
  1. the set of unobservable state variables $X_t$
  2. the set of observable evidence variables $E_t$

• The observation at time $t$ is $E_t = e_t$ for some set of values $e_t$

• $X_{a:b}$ denotes the set of variables from $X_a$ to $X_b$
**Transition and Sensor Models**

- **Transition model**
  - Models how the world changes over time
  - Specifies a probability distribution
    - Over state variables at time \( t \)
    - Given values at previous times

- **Sensor model**
  - Models how evidence gets its values (sensor data)
    - E.g.: BloodSugar\(_t\) → MeasuredBloodSugar\(_t\)

**Markov Assumption**

- **Markov Assumption:**
  - \( X_t \) depends on some finite (usually fixed) number of previous \( X_i \)'s

- **First-order Markov process:** \( P(X_t | X_{0:t-1}) = P(X_t | X_{t-1}) \)
  - \( k^{th} \) order: depends on previous \( k \) time steps

- **Sensor Markov assumption:** \( P(E_t | X_{0:t}, E_{0:t-1}) = P(E_t | X_t) \)
  - Agent's observations depend *only* on the actual current state of the world
Stationary Process

• Infinitely many possible values of $t$
  • Does each timestep need a distribution?

• Assume stationary process:
  • Changes in the world state are governed by laws that do not themselves change over time
  • Transition model $P(X_t | X_{t-1})$ and sensor model $P(E_t | X_t)$ are time-invariant, i.e., they are the same for all $t$

Complete Joint Distribution

• Given:
  • Transition model: $P(X_t | X_{t-1})$
  • Sensor model: $P(E_t | X_t)$
  • Prior probability: $P(X_0)$

• Then we can specify complete joint distribution of a sequence of states:

$$P(X_0, X_1, ..., X_t, E_1, ..., E_t) = P(X_0) \prod_{i=1}^{t} P(X_i | X_{i-1}) P(E_i | X_i)$$
Example

\[
\begin{array}{c|c|c}
R_{t-1} & P(R_t|R_{t-1}) & \\
\hline
T & 0.7 & \\
F & 0.3 & \\
\end{array}
\]

This should look like a finite state automaton (since it is one)

Inference Tasks

- **Filtering** or monitoring: \(P(X_t|e_1,\ldots,e_t)\)
  Compute the current belief state, given all evidence to date

- **Prediction**: \(P(X_{t+k}|e_1,\ldots,e_t)\)
  Compute the probability of a future state

- **Smoothing**: \(P(X_k|e_1,\ldots,e_t)\)
  Compute the probability of a past state (hindsight)

- **Most likely explanation**: 
  \[\arg\max_{x_1,\ldots,x_T} P(x_1,\ldots,x_T|e_1,\ldots,e_t)\]
  Given a sequence of observations, find the sequence of states that is most likely to have generated those observations
Examples

- **Filtering**: What is the probability that it is raining today, given all of the umbrella observations up through today?

- **Prediction**: What is the probability that it will rain the day after tomorrow, given all of the umbrella observations up through today?

- **Smoothing**: What is the probability that it rained yesterday, given all of the umbrella observations through today?

- **Most likely explanation**: If the umbrella appeared the first three days but not on the fourth, what is the most likely weather sequence to produce these umbrella sightings?

Filtering

- Maintain a current state estimate and update it
  - Rather than looking at all percepts (observed values) in history
  - So, given result of filtering up to \( t \), compute \( t+1 \) from \( e_{t+1} \)

- We use **recursive estimation** to compute
  \( P(X_{t+1} \mid e_{1:t+1}) \) as a function of \( e_{t+1} \) and \( P(X_{t} \mid e_{1:t}) \)

- We can write this as:
  \[
P(X_{t+1} \mid e_{1:t+1}) = P(X_{t+1} \mid e_{1:t}, e_{t+1})
  \]
Filtering 2

• $P(X_{t+1} \mid e_{1:t+1})$ as a function of $e_{t+1}$ and $P(X_t \mid e_{1:t})$

\[
P(X_{t+1} \mid e_{1:t+1}) = P(X_{t+1} \mid e_{t+2}, e_{t+1}) = \alpha P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{t+2}) = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{t+2})
\]

• This leads to a recursive definition:

\[
f_{1:t+1} = \alpha \text{ FORWARD } (f_{1:t}, e_{t+1})
\]

Filtering Example

\[
P(X_{t+1} \mid e_{1:s+1}) = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:s})
\]

<table>
<thead>
<tr>
<th>$R_{t+1}$</th>
<th>$P(R_{t+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0.7</td>
</tr>
<tr>
<td>F</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$R_t$</th>
<th>$P(U_t \mid R_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0.9</td>
</tr>
<tr>
<td>F</td>
<td>0.2</td>
</tr>
</tbody>
</table>

What is the probability of rain on Day 2, given a uniform prior of rain on Day 0, $U_1 = \text{true}$, and $U_2 = \text{true}$?
Decision Making Under Uncertainty

• Many environments have multiple possible outcomes
• Some of these outcomes may be good; others may be bad
• Some may be very likely; others unlikely
• What’s a poor agent to do?
Reasoning Under Uncertainty

• So how do we do reasoning under uncertainty and with inexact knowledge?
  • Heuristics
    • Mimic heuristic knowledge processing methods used by experts
  • Empirical associations
    • Experiential reasoning
    • Based on limited observations
  • Probabilities
    • Objective (frequency counting)
    • Subjective (human experience)

Non-deterministic vs. Probabilistic Uncertainty

{a,b,c} → decision that is best for worst case
Non-deterministic model

{a(p_a), b(p_b), c(p_c)} → decision that maximizes expected utility value
Probabilistic model

~ Adversarial search
Decision Theory

- Combine **probability** and **utility**

→ Agent that makes **rational** decisions
  - On average, lead to desired outcome

- Immediate simplifications:
  - Want most desirable immediate outcome (episodic)
  - Nondeterministic, partially observable world

- Definition: result of an action $a$ leads to outcome $s'$:
  - $\text{RESULT}(a)$ is a random variable; domain is possible outcomes
  - $P(\text{RESULT}(a) = s' \mid a, e)$

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Expected Utility

- Goal: find best expected outcome

- Random variable $X$ with:
  - $n$ values $x_1, \ldots, x_n$
  - Distribution $(p_1, \ldots, p_n)$

- $X$ is the state reached after doing an action $A$ under uncertainty

- Utility function $U(s)$ is the utility of a state, i.e., **desirability**
Expected Utility

- X is state reached after doing an action A under uncertainty
- $U(s)$ is the utility of a state $\leftarrow$ desirability
- The expected utility of action A, given evidence $EU(a|e)$, is average utility of outcomes (states in S), weighted by probability an action occurs:

$$EU[A] = \sum_{i=1}^{n} p(x_i|A)U(x_i)$$

One State/One Action Example

$$U(A1, S0) = 100 \times 0.2 + 50 \times 0.7 + 70 \times 0.1$$

$$= 20 + 35 + 7$$

$$= 62$$
One State/Two Actions Example

- $U(A_1, S_0) = 62$
- $U(A_2, S_0) = 74$
- $U(S_0) = \max_a \{U(a, S_0)\} = 74$

Introducing Action Costs

- $U(A_1, S_0) = 62 - 5 = 57$
- $U(A_2, S_0) = 74 - 25 = 49$
- $U(S_0) = \max_a \{U(a, S_0)\} = 57$
MEU Principle

• A rational agent should choose the action that maximizes agent’s expected utility

• This is the basis of the field of decision theory

• The MEU principle provides a normative criterion for rational choice of action

• …AI is solved!

Not Quite…

• Must have a complete model of:
  • Actions
  • Utilities
  • States

• Even if you have a complete model, decision making is computationally intractable

• In fact, a truly rational agent takes into account the utility of reasoning as well (bounded rationality)

• Nevertheless, great progress has been made in this area
  • We are able to solve much more complex decision-theoretic problems than ever before
Axioms of Utility Theory

• Orderability
  \[(A>B) \lor (A<B) \lor (A\sim B)\]

• Transitivity
  \[(A>B) \land (B>C) \Rightarrow (A>C)\]

• Continuity
  \[A>B>C \Rightarrow \exists p \ [p,A; 1-p,C] \sim B\]

• Substitutability
  \[A\sim B \Rightarrow [p,A; 1-p,C] \sim [p,B; 1-p,C]\]

• Monotonicity
  \[A>B \Rightarrow (p \geq q \iff [p,A; 1-p,B] \succ [q,A; 1-q,B])\]

• Decomposability
  \[ [p,A; 1-p, [q,B; 1-q, C]] \sim [p,A; (1-p)q, B; (1-p)(1-q), C] \]

Money Versus Utility

• Money \text{n} Utility
  \[\text{More money is better, but not always in a linear relationship to the amount of money}\]

• Expected Monetary Value

• Risk-averse: \(U(L) < U(S_{EMV(L)})\)

• Risk-seeking: \(U(L) > U(S_{EMV(L)})\)

• Risk-neutral: \(U(L) = U(S_{EMV(L)})\)
Value Function

- Provides a ranking of alternatives, but not a meaningful metric scale
- Also known as an “ordinal utility function”
- Sometimes, only relative judgments (value functions) are necessary
- At other times, absolute judgments (utility functions) are required