Quadric Error Metrics

To do
- Continue to work on ray programming assignment
- Continue to think about final project
- Next Monday: Dr. Lee Butler’s guest lecture

Background: quadrics
- A quadrics are all surfaces that can be expressed as a second degree polynomial in x, y, and z.

Quadric surfaces
- General implicit form
  \[ ax^2 + 2bxy + 2cz + 2dxz + ey^2 + 2fyz + 2gxy + hw^2 + 2iwx + jw^2 = 0. \]
- Matrix form
  - Setting w to 1, this provides the ability to position the quadrics in space
  \[ \mathbf{x}'^TQ\mathbf{x}' = 0, \]
  where \( \mathbf{x}' = \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} \) and \( Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \).
Surface Simplification: Goals

- Efficiency (70000 to 100 faces in 15s in 1997)
- High quality, feature preserving (primary appearance emphasized rather than topology)
- Generally, non-manifold models, collapse disjoint regions

7809 tris 3905 tris 488 tris

Background: Manifold

- Mathematical term of a surface, all of those points have a neighborhood which is topologically equivalent to a disk.
- A manifold with boundary is a surface all of whose points have either a disk or a half-disk.
- A polygonal surface is a manifold (with boundary) if every edge has exactly two incident faces, and the neighborhood of every vertex consists of a closed loop of faces (or a single fan of faces on the boundary)

Overview and Resources

- Garland and Heckbert SIGGRAPH 97 paper
  - Greedy decimation algorithm
  - Pair collapse (allow edge + non-edge collapse)
- Evaluate potential collapse
- Determine optimal new vertex locations
- Garland website, implementation notes in his thesis
  - http://mgarland.org/research/quadrics.html
- Notes in this and the previous lecture

Algorithm Outline

- Restrict process to a set of valid pairs:
  - $(v_i, v_j)$ is an edge, or
  - $|v_i - v_j| < t$, a threshold
  - $t = 0$ restricts to edge contraction
  - $t > 0$ can connect distant regions or yield $O(n^2)$ pairs
- Iteratively remove best pair and update valid pairs list:
  - Each vertex has a set with the pairs it belongs to:
    - $v_i \rightarrow \text{Pairs}(v_i)$
    - $(v_i, v_j) \rightarrow v = \text{Pairs}(v) = \text{Pairs}(v_i) \cup \text{Pairs}(v_j)$
  - But how to choose best pair?
Choose best pair
- Based on point-to-plane distance
- Better quality than point-to-point

Background: Computing Planes
- Each triangle in mesh has associated plane $ax + by + cz + d = 0$
- For a triangle, find its (normalized) normal using cross products
  \[
  \overrightarrow{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{||\overrightarrow{AB} \times \overrightarrow{AC}||} \quad \overrightarrow{n} \cdot \overrightarrow{v} = 0
  \]
- Plane equation:
  \[
  \overrightarrow{n} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad d = -\overrightarrow{A} \cdot \overrightarrow{v}
  \]

Quadric Error Metrics
- Sum of squared distances from vertex to planes
  \[
  \Delta = \sum_{p} \text{Dist}(v,p)^2 \\
  \text{Dist}(v,p) = ax + by + cz + d = p^T \overrightarrow{v}
  \]
- Common mathematical trick: Quadratic form = symmetric matrix $Q$ multiplied twice by a vector.
But What Are These Quadrics Really Doing?

- Almost always ellipsoids
  - When Q is positive definite
- Characterize error at vertex
  - Vertex at center of each ellipsoid
  - Move it anywhere on ellipsoid with constant error
- Capture local shape of surface
  - Stretch in least curved direction

Quadric Visualization

- Ellipsoids: iso-error surfaces
- Smaller ellipsoid = greater error for a given motion
- Lower error for motion parallel to surface
- Lower error in flat regions than at corners
- Elongated in "cylindrical" regions

Using Quadrics

- Approximate error of edge collapses
  - Each vertex v has associated quadric Q
  - Error of collapsing v₁ and v₂ to v' is v'^T Q₁ v' + v'^T Q₂ v'
  - Quadric for new vertex v' is Q' = Q₁ + Q₂

Find optimal location v' after collapse
Algorithm Summary

- Compute the Q matrices for all the initial vertices
- Select all valid pairs
- Compute the optimal contraction target \( v \) for each valid pair. The error \( v'(Q_1 + Q_2) v' \) of this target vertex becomes the cost of contracting that pair.
- Place all pairs in a heap keyed on cost with minimum cost pair on the top
- Iteratively remove the least cost pair, contract this pair, and update the costs of all valid pairs of interest

Results

<table>
<thead>
<tr>
<th>Original</th>
<th>Quadrics</th>
<th>1000 tris</th>
<th>100 tris</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>Quadrics</td>
<td>250 tris</td>
<td>250 tris, edge collapses only</td>
</tr>
</tbody>
</table>

Additional Details

- Preserving boundaries / discontinuities (weight quadrics by appropriate penalty factors)
- Preventing mesh inversion (flipping of orientation): compare normal of neighboring faces, before after
- Has been modified for many other applications
  - E.g., in silhouettes, want to make sure volume always increases, never decreases
  - Take color and texture into account (follow up paper)

Implementation Tips

- Incremental, test, debug, simple cases
- Find good data structure for heap etc.
- May help to visualize error quadrics if possible
- Challenging, but hopefully rewarding assignment