Propositional Logic

Chapter 7.4–7.7

Some material adopted from notes by Andreas Geyer-Schulz and Chuck Dyer

Propositional logic syntax

- Users specify
  - Set of propositional symbols (e.g., P, Q) whose values can be **True** or **False**
  - What each *means*, e.g.: P: *"It's hot"*, Q: *"It's humid"*
- A sentence (well formed formula) is defined as:
  - Any symbol is a sentence
  - If S is a sentence, then **¬S** is a sentence
  - If S is a sentence, then **(S)** is a sentence
  - If S and T are sentences, then so are **(S ∨ T), (S ∧ T), (S → T),** and **(S ↔ T)**
  - A finite number of applications of the rules

Examples of PL sentences

- **Q**

  "It's humid"
- **$Q \rightarrow P$**

  "If it's humid, then it's hot"
- **$(P \land Q) \rightarrow R$**

  "If it's hot and it's humid, then it's raining"
- We're free to choose better symbols, e.g.:

  Hot for "It's hot"

  Humid for "It's humid"

  Raining for "It's raining"

Some terms

- Given the truth values of all symbols in a sentence, it can be ***evaluated*** to determine its **truth value** (True or False)
- We consider a **Knowledge Base** (KB) to be a set of sentences that are all True
- A **model** for a KB is a **possible world** – an assignment of truth values to propositional symbols that makes each KB sentence true

A simple example

The KB

P

$Q \lor \neg R$

The KB has 2 sentences.

The KB has 3 variables.

Models for the KB

P	Q	R	
T	T	F	#1
T	T	T	#2
T	F	F	#3

The KB has 3 models. Each model has a value for every variable in the KB such every sentence evaluates to true.

Another simple example

The KB

$P \wedge Q$

$R \wedge \neg P$

The KB has 2 sentences.

The KB has 3 variables.

Models for the KB

P	Q	R

The KB has no models. There is no assignment of True or False to every variable that makes every sentence in the KB true

More terms

- A **valid sentence** or **tautology:** one that’s **True** under all interpretations, no matter what the world is actually like or what the semantics is. Example: “It's raining or it's not raining” $(P \lor \neg P)$

- An **inconsistent sentence** or **contradiction**: a sentence that’s **False** under all interpretations. The world is never like what it describes, as in “It's raining and it's not raining.” $(P \land \neg P)$

Truth tables

Used to define meaning of logical connectives

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

Given a value for P and for Q, the truth table defines the value of P∨Q

Truth tables

Used to define meaning of logical connectives **and to determine when a complex sentence is true given values of its symbols**

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

The value of complex sentences can be determined from the values of their elements

Example of a truth table used for a complex sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	False	False	False	True
False	True	True	False	True
True	False	True	True	True
True	True	True	False	True

The implies connective: P → Q

→ is a *logical connective*

- *P* → *Q* is a **logical sentence** and has a truth value, i.e., is either **True** or **False**
- If the sentence is in a KB, it can be used by a rule (*Modes Ponens*) to infer that Q is True if P is True in the KB
- Given a KB where P=True and Q=True, we can derive/infer/prove that P→Q is True
- Note: P→ Q is equivalent to ~P∨Q

$P \rightarrow Q$

When is $P \rightarrow Q$ true? Check all that apply

- ❑ P=Q=true
- ❑ P=Q=false
- ❑ P=true, Q=false
- ❑ P=false, Q=true

$P \rightarrow Q$

When is *P→Q* true? Check all that apply

- ☑ P=Q=true
- ☑ P=Q=false
- ☐ P=true, Q=false
- ☑ P=false, Q=true

- We can get this from the truth table for $\rightarrow$
- Note: in FOL it's much harder to prove that a conditional true, e.g., prime(x) $\rightarrow$ odd(x)

  you must prove it's true for every possible value of x

$P \rightarrow Q \equiv {\sim}P \vee Q$

- $P \rightarrow Q$ is equivalent to ${\sim}P \vee Q$
- We can show this by looking at a truth table

P	Q	$P \rightarrow Q$	${\sim}P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

These two columns are the same

Models for a KB

- KB: [PvQ, P→R, Q→R]
- What are the sentences?

  s1: PVQ
  s2: P→R
  s3: Q→R
- What are the propositional variables?

  P, Q, R
- What are the candidate models?
  1) Consider all **eight** possible assignments of T|F to P, Q, R
  2) Check if each sentence is consistent with the model

P	Q	R	s1	s2	s3
F	F	F	X	✓	✓
F	F	T	X	✓	✓
F	T	F	✓	✓	X
F	T	T	✓	✓	✓
T	F	F	✓	X	✓
T	F	T	✓	✓	✓
T	T	F	✓	X	X
T	T	T	✓	✓	✓

Here X means the model makes the sentence False and ✓ means it doesn't make it False

Models for a KB

- KB: [PvQ, P→R, Q→R]
- What are the sentences?

  s1: PVQ

  s2: P→R

  s3: Q→R
- What are the propositional variables?

  P, Q, R
- What are the candidate models?
  1) Consider all possible assignments of T|F to P, Q, R
  2) Check truth tables for consistency, eliminate a row that does not make every KB sentence true

	P	Q	R	s1	s2	s3
	~~F~~	~~F~~	~~F~~	~~X~~	~~✓~~	~~✓~~
	~~F~~	~~F~~	~~T~~	~~X~~	~~✓~~	~~✓~~
	~~F~~	~~T~~	~~F~~	~~✓~~	~~✓~~	~~X~~
➡	F	T	T	✓	✓	✓
	~~T~~	~~F~~	~~F~~	~~✓~~	~~X~~	~~✓~~
➡	T	F	T	✓	✓	✓
	~~T~~	~~T~~	~~F~~	~~✓~~	~~X~~	~~X~~
➡	T	T	T	✓	✓	✓

- Only 3 models are consistent with KB
- R true in **all** of them
- Therefore, R is true and can be added to KB

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