Reasoning with Propositional Logic

Chapter 7.4–7.8

Some material adopted from notes by Andreas Geyer-Schulz and Chuck Dyer

Overview

- There are many ways to approach reasoning with propositional logic
- We'll look at one, ***resolution refutation***, that can be extended to first order logic
- Later, we will look other approaches that are special to propositional logic
  - Some of there are more efficient

Reasoning / Inference

- **Logical inference** creates new sentences that logically follow from a set of sentences, i.e., those in the KB
- It can also detect if a KB is inconsistent, i.e., has sentences that entail a **contradiction**
- An inference rule is **sound** if every sentence it produces from a KB logically follows from the KB
  - i.e., inference rule creates no contradictions
- An inference rule is **complete** if it can produce every expression that logically follows from (is entailed by) the KB

Sound rules of inference

Examples of sound rules of inference

Each can be shown to be sound using a truth table

RULE	PREMISE	CONCLUSION
Modus Ponens	$A, A \rightarrow B$	B
And Introduction	A, B	$A \wedge B$
And Elimination	$A \wedge B$	A
Double Negation	$\neg\neg A$	A
Unit Resolution	$A \vee B, \neg B$	A
Resolution	$\mathbf{A \vee B, \neg B \vee C}$	$\mathbf{A \vee C}$

Resolution

- **Resolution** is a valid inference rule producing a new clause implied by two clauses containing *complementary literals*

  Literal: atomic symbol or its negation, i.e., P, ~P

  complementary literals: any variable and its negation

- Amazingly, this is the **only interference rule needed** to build a sound & complete theorem prover
  - Based on proof by contradiction, usually called resolution refutation
- This property of the resolution rule was found by Alan Robinson (CS, U. of Syracuse) in the mid 1960s

Resolution

- A KB is a set of sentences all of which are true, i.e., a conjunction of sentences
- To use resolution, put KB into **conjunctive normal form** (CNF)
  - Each sentence is a disjunction of one or more literals (positive or negative atoms)
- Every KB can be put into CNF, by rewriting its sentences using standard tautologies, e.g.:
  - $P \rightarrow Q \equiv \sim P \vee Q$
  - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \equiv (P \vee Q), (P \vee R)$

Resolution Example

Tautologies

(A→B) ↔ (~A ∨ B)

(A∨ (B ∧ C)) ↔ (A∨B)∧(A∨C)

- KB: [P→Q , Q→R∧S]
- KB: [P→Q , Q→R, Q→S]
- KB in CNF: [~P∨Q , ~Q∨R , ~Q∨S]
- Resolve KB[0] & KB[1] producing

  KB[4]: ~P∨R *(i.e., P→R)*
- Resolve KB[0] & KB[2] producing

  KB[5]: ~P∨S *(i.e., P→S)*
- New KB:

  [~P∨Q , ~Q∨R, ~Q∨S, ~P∨R, ~P∨S]

0: ~P∨Q
1: ~Q∨R
2: ~Q∨S
3: ~P∨R
4: ~P∨S

Proving it's raining with rules

- A **proof** is a sequence of sentences, where each is a premise (i.e., a given) or is derived from earlier sentences in the proof by an inference rule
- Last sentence is the **theorem** (also called goal or query) that we want to prove
- The *weather problem* using traditional reasoning

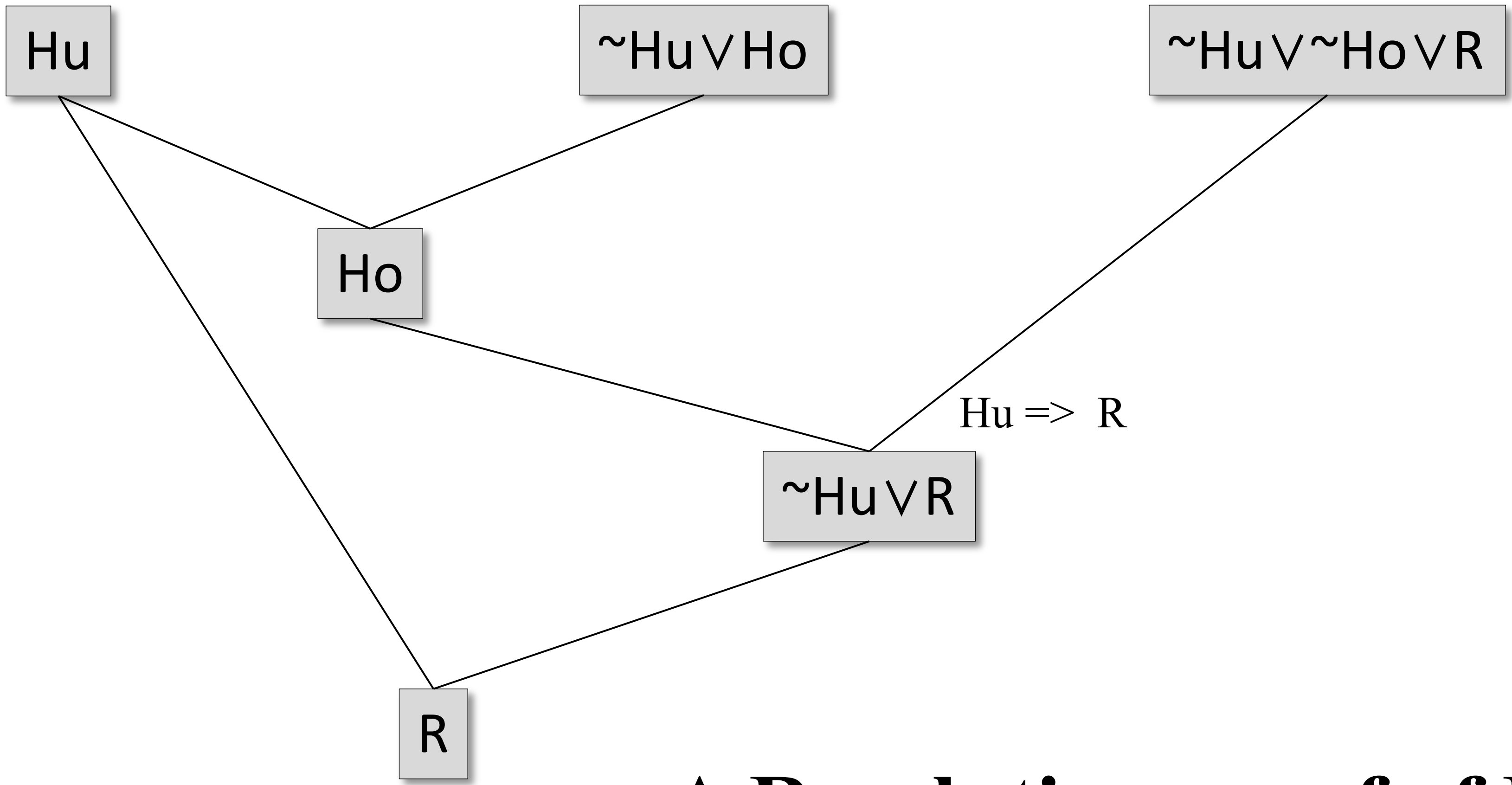
1 Hu	premise	"It's humid"
2 $Hu \rightarrow Ho$	premise	"If it's humid, it's hot"
3 Ho	modus ponens(1,2)	"It's hot"
4 $(Ho \wedge Hu) \rightarrow R$	premise	"If it's hot & humid, it's raining"
5 $Ho \wedge Hu$	and introduction(1,3)	"It's hot and humid"
6 R	modus ponens(4,5)	"It's raining"

Proving it's raining with resolution

Hu

Hu => Ho
~Hu ∨ Ho

Hu ∧ Ho => R
~(Hu ∧ Ho) ∨ R
~Hu ∨ ~Ho ∨ R



A Resolution proof of R

A simple proof procedure

This procedure generates new sentences in a KB

1. Convert all sentences in the KB to CNF[1]
2. Find all pairs of sentences with *complementary literals*[2] that have not yet been resolved
3. If there are no pairs stop else resolve each pair, adding the result to the KB and go to 2

- Is it sound?, complete? always terminate?

1: a KB in conjunctive normal form is a set of disjunctive sentences

2: a literal is a variable or its negation

Propositional Resolution

- It is sound!
- It’s not *generatively complete* in that it can’t derive all clauses that follow from the KB
  - The issues are not serious limitations, though
  - Example: if the KB includes P and includes Q we won’t derive P ^ Q
- It will always terminate
- But generating all clauses that follow can take a long time and many may be useless

Refutation proofs

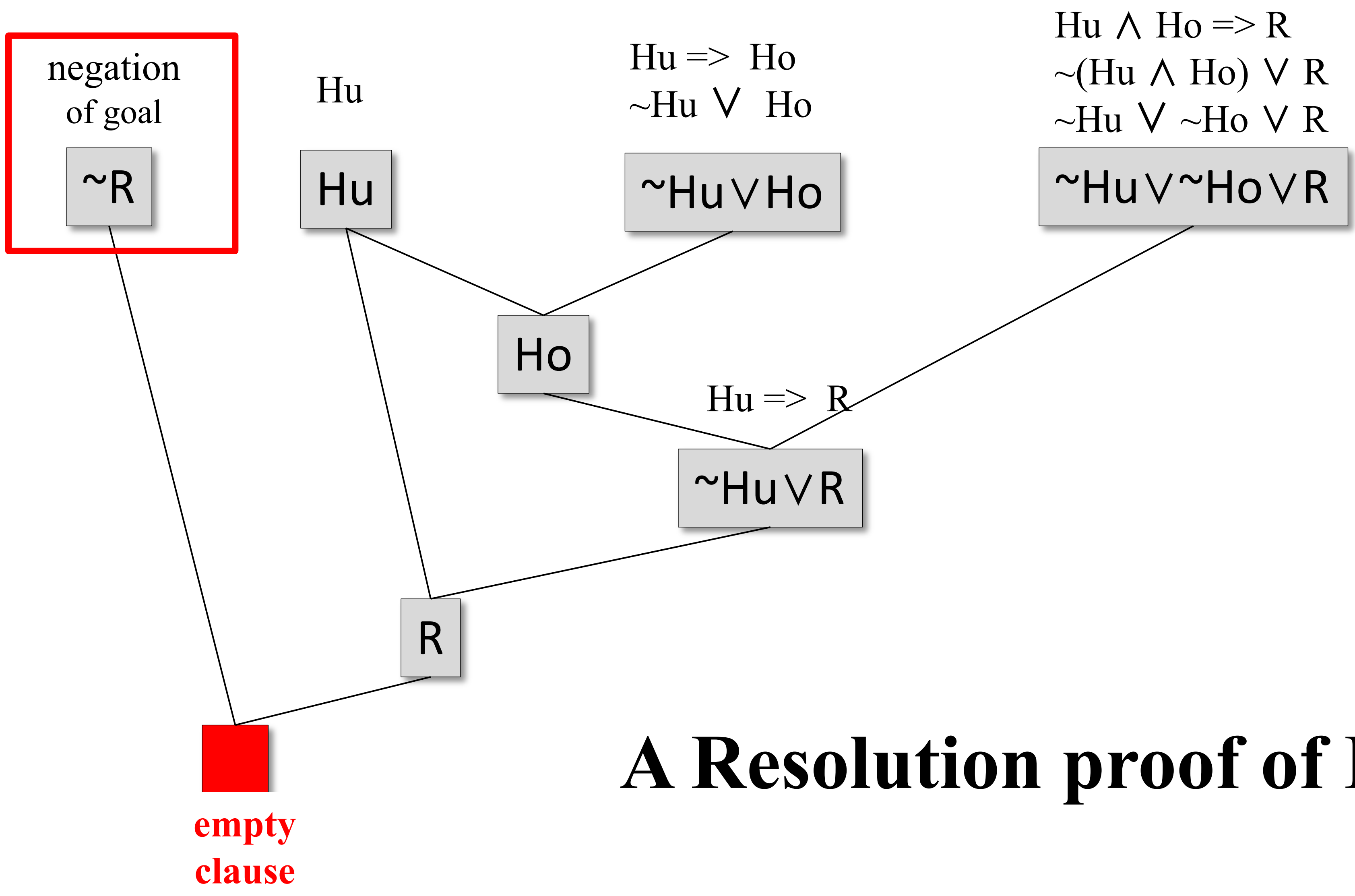
- Common use case: we have a question/goal (e.g, **P**) and want to know if it's true given our KB
- We assume every sentence in our initial KB is true
- A refutation proof is a common approach:
  - We start with a KB with all true facts
  - Add negation of what we want to prove (e.g., **~P**)
  - Try to find a **contradiction**
  - If our proof ever produces one, it must be due to adding **~P**, so goal is proven
- Procedure easy to focus & control, so is tends to be more efficient

Resolution refutation proof of P

1. Add negation of goal to the KB, **~P**
2. Convert all sentences in KB to CNF
3. Find pairs of sentences with complementary literals that have not yet been resolved
4. If there are no pairs stop else resolve each pair, adding the result to the KB and go to 2

- If we get an empty clause (i.e., contradiction) then **P** follows from the KB
  - e.g., resolving **X** with **~X** results in an empty clause
- If not, conclusion can't be proved from the KB

Proving it's raining with refutation resolution



A Resolution proof of R

Propositional Reasoning

- Many other reasoning tasks with propositions
- **Boolean Satisfiability** (SAT) involves finding a set of values that will make a KB true
  - While NP complete, heuristic SAT-algorithms can solve problems with 10s of thousands of variables
- There are many efficient and scalable proof procedures for sets of propositions
- Reducing a problem to a set of propositions and using an efficient proof technique is often a good way to solve a problem

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