Basic Finite Element Method

The purpose of this lecture is to show a systematic method of developing a computer program to find the numerical solution for a partial differential equation of fourth order in four dimensions with known boundary values using a generic finite element method.

Be advised that most numerical solution of partial differential equations makes use of special techniques for specific problems. There are layer upon layer of approximation techniques and implementation variations. Many implementations have evolved and the details of the current software design are not available and may not be understood.

One method for numerically solving partial differential equations is the Finite Element Method, FEM, and specifically the Galerkin method. Partial differential equation is used here to include ordinary differential equation in order to use one notation for all orders and all numbers of dependent variables. The independent analytic variable will be u and the dependent variables will be x, y, z and t for first through fourth dimensions. The numerical solution will be $U(x_i, y_j, z_k, t_w)$ values at discrete points, not necessarily uniformly spaced, in the problems dimensions.

The development will proceed, beginning with the one dimensional case, onto the four dimensional case.

Given a one dimensional linear partial differential equation with dependent variable u and independent variable x

$$L\left(u(x)\right) = f(x)$$

L is a general linear differential operator of some specific order. Samples for up to fourth order in up to four dimensions are shown below.

Find the approximate solution $U(x)$ numerically, that has boundary conditions chosen that make the problem of finding $u(x)$ well posed on the domain Ω for all $x \in \Omega$. We approximate $U(x)$

$$U(x) = \sum_{j=1}^{n} U_j \phi_j(x)$$

where $\phi_j(x)$ is the j_{th} polynomial in a set of orthogonal polynomials such that:

$$\phi_j(x_i) = \begin{cases} 1 \ for \ i = j \\ 0 \ otherwise \end{cases}$$

We take $j = 1 \ldots n$ for specific nodes $x_1, x_2, \ldots, x_n$.

We denote the approximated solution $U(x_j)$ as U_j at node x_j.

We will use ϕ as a set of Lagrange orthogonal polynomials.

The Galerkin Method states:

$$\int_\Omega L\left(U(x)\right)\phi_i(x)dx = \int_\Omega f(x)\phi_i(x)dx$$

Substituting for $U(x)$ yields

$$\int_\Omega L\left(\sum_{j=1}^{n} U_j \phi_j(x)\right)\phi_i(x)dx = \int_\Omega f(x)\phi_i(x)dx$$

Bringing the summation out of the integral yields

$$\sum_{j=1}^{n} U_j \int_{\Omega} L\left(\phi_j(x)\right) \phi_i(x) dx = \int_{\Omega} f(x) \phi_i(x) dx$$

$L(\phi_j(x))$ means a substitution in $L(u(x))$ where $u(x)$ becomes $\phi_j(x)$, $u'(x)$ becomes $\phi_j^{'}(x)$, and $u''(x)$ becomes $\phi_j^{''}(x)$ etc.

Writing the above in matrix form using the index i for rows and index j for columns yields

$$\begin{vmatrix} \int_\Omega L\left(\phi_1(x)\right)\phi_1(x)dx & \int_\Omega L\left(\phi_2(x)\right)\phi_1(x)dx & \dots & \int_\Omega L\left(\phi_n(x)\right)\phi_1(x)dx \\ \int_\Omega L\left(\phi_1(x)\right)\phi_2(x)dx & \int_\Omega L\left(\phi_2(x)\right)\phi_2(x)dx & \dots & \int_\Omega L\left(\phi_n(x)\right)\phi_2(x)dx \\ & & \dots & \\ \int_\Omega L\left(\phi_1(x)\right)\phi_n(x)dx & \int_\Omega L\left(\phi_2(x)\right)\phi_n(x)dx & \dots & \int_\Omega L\left(\phi_n(x)\right)\phi_n(x)dx \end{vmatrix} \times \begin{vmatrix} U_1 \\ U_2 \\ \dots \\ U_n \end{vmatrix} = \begin{vmatrix} \int_\Omega f(x)\phi_1(x)dx \\ \int_\Omega f(x)\phi_2(x)dx \\ \dots \\ \int_\Omega f(x)\phi_n(x)dx \end{vmatrix}$$

Note that the above applies for the "internal" non boundary nodes.

Given Dirichlet boundary values v_1 at x_1 and v_n at x_n the first and last rows of the above matrix equation would be:

$$\begin{vmatrix} 1\ 0\ \dots\ 0 \\ \dots \\ 0\ 0\ \dots\ 1 \end{vmatrix} \times \begin{vmatrix} U_1 \\ \dots \\ U_n \end{vmatrix} = \begin{vmatrix} v_1 \\ \dots \\ v_n \end{vmatrix}$$

The boundary rows may be eliminated and a $(n-2)$ system of equations are solved to find the U_j for $j = 2 \dots n-1$

Note that, in general, numerical integration is required to compute the matrix elements and the right hand side vector elements.

Given a two dimensional linear partial differential equation with dependent variable u and independent variables x, y

$$L\left(u(x,y)\right) = f(x,y)$$

L is a general linear differential operator of some specific order. Samples for up to fourth order in up to four dimensions are shown below.

Find the approximate solution $U(x,y)$ numerically, that has boundary conditions chosen that make the problem of finding $u(x,y)$ well posed on the domain Ω for all $(x,y) \in \Omega$.

We denote the approximated solution $U(x_i, y_j)$ as U_k at node (x_i, y_j).

We take $i = 1 \ldots nx$ for specific nodes $x_1, x_2, \ldots, x_{nx}$ and $j = 1 \ldots ny$ for specific nodes $y_1, y_2, \ldots, y_{ny}$.

Thus, without loss of generality, k runs over all pairs (x_i, y_j) $k = (i-1) \times ny + j$ and $nxy = nx \times ny$.

Let $U(x,y) = \sum_{k=1}^{nxy} U_k \phi_k(x,y)$

We will use

$$\phi_k(x,y) = \phi_i(x)\phi_j(y)$$

where $\phi_k(x,y)$ is the k_{th} polynomial in a set of two dimensional orthogonal polynomials such that:

$$\phi_k(x_i, y_j) = \begin{cases} 1 \; for \; i = k, j = k \\ 0 \; otherwise \end{cases}$$

We will use ϕ as a set of Lagrange orthogonal polynomials.

The Galerkin Method states:

$$\int_\Omega L\left(U(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

Substituting for $U(x,y)$ yields

$$\int_\Omega L\left(\sum_{k=1}^{nxy} U_k \phi_k(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

Bringing the summation out of the integral yields

$$\sum_{k=1}^{nxy} U_k \int_\Omega L\left(\phi_k(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

$L(\phi_k(x,y))$ means a substitution in $L(u(x,y))$ where $u(x,y)$ becomes $\phi_k(x,y)$, $ux(x,y)$ becomes $\phi_i^{'}(x,y)\,\phi_j(x,y)$, $uy(x,y)$ becomes $\phi_i(x,y)\,\phi_j^{'}(x,y)$, $uxy(x,y)$ becomes $\phi_i^{'}(x,y)\,\phi_j^{'}(x,y)$, $uxx(x,y)$ becomes $\phi_i^{''}(x,y)\,\phi_j(x,y)$, $uyy(x,y)$ becomes $\phi_i(x,y)\,\phi_j^{''}(x,y)$, etc.

Writing the above in matrix form using the index $k = (i-1) \times ny + j$ for rows and index $l = (i-1) \times ny + j$ for columns yields

$$\left| \begin{array}{cccc} \int_\Omega L\left(\phi_1(x,y)\right)\phi_1(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_1(x,y)dx\,dy & \ldots & \int_\Omega L\left(\phi_{nxy}(x,y)\right)\phi_1(x,y)dx\,dy \\ \int_\Omega L\left(\phi_1(x,y)\right)\phi_2(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_2(x,y)dx\,dy & \ldots & \int_\Omega L\left(\phi_{nxy}(x,y)\right)\phi_2(x,y)dx\,dy \\ & & \ldots & \\ \int_\Omega L\left(\phi_1(x,y)\right)\phi_{nxy}(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_{nxy}(x,y)dx\,dy & \ldots & \int_\Omega L\left(\phi_{nxy}(x,y)\right)\phi_{nxy}(x,y)dx\,dy \end{array} \right| \times$$

$$\left| \begin{array}{c} U_1 \\ U_2 \\ \ldots \\ U_{nxy} \end{array} \right| = \left| \begin{array}{c} \int_\Omega f(x,y)\phi_1(x,y)dx\,dy \\ \int_\Omega f(x,y)\phi_2(x,y)dx\,dy \\ \ldots \\ \int_\Omega f(x,y)\phi_{nxy}(x,y)dx\,dy \end{array} \right|$$

Note that the above applies for the “internal” non boundary nodes.

Given Dirichlet boundary values, e.g. v_1 at (x_1, y_1) and v_{nxy} at (x_{nx}, y_{ny}) the first and last rows of the above matrix equation would be:

$$\left| \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & 1 \end{array} \right| \times \left| \begin{array}{c} U_1 \\ \ldots \\ U_{nxy} \end{array} \right| = \left| \begin{array}{c} v_1 \\ \ldots \\ v_{nxy} \end{array} \right|$$

The many boundary rows may be eliminated and a $(nx-2)(ny-2)$ system of equations are solved to find the U_k for $i = 2 \ldots nx-1, j = 2 \ldots ny-1$

Note that, in general, numerical integration is required to compute the matrix elements and the right hand side vector elements.

Given a three dimensional linear partial differential equation with dependent variable u and independent variables x, y, z

$$L\left(u(x, y, z)\right) = f(x, y, z)$$

L is a general linear differential operator of some specific order. Samples for up to fourth order in up to four dimensions are shown below.

Find the approximate solution $U(x, y, z)$ numerically, that has boundary conditions chosen that make the problem of finding $u(x, y, z)$ well posed on the domain Ω for all $(x, y, z) \in \Omega$.

We denote the approximated solution $U(x_i, y_j, z_k)$ as U_m at node (x_i, y_j, z_k).

We take $i = 1 \ldots nx$ for specific nodes $x_1, x_2, \ldots, x_{nx}$ and $j = 1 \ldots ny$ for specific nodes $y_1, y_2, \ldots, y_{ny}$. $k = 1 \ldots nz$ for specific nodes $z_1, z_2, \ldots, z_{nz}$.

Thus, without loss of generality, m runs over all (x_i, y_j, z_k) $m = (i-1) \times ny \times nz + (j-1) \times nz + k$ and $nxyz = nx \times ny \times nz$.

Let $U(x, y, z) = \sum_{m=1}^{nxyz} U_m \phi_m(x, y, z)$

We will use

$$\phi_m(x, y, z) = \phi_i(x)\phi_j(y)\phi_k(z)$$

where $\phi_m(x, y, z)$ is the m_{th} polynomial in a set of three dimensional orthogonal polynomials such that:

$$\phi_m(x_i, y_j, z_k) = \begin{cases} 1 \; for \; i = m, j = m, k = m \\ 0 \; otherwise \end{cases}$$

We will use ϕ as a set of Lagrange orthogonal polynomials.

The Galerkin Method states:

$$\int_{\Omega} L\left(U(x, y, z)\right) \phi_l(x, y, z) dx\, dy\, dz = \int_{\Omega} f(x, y, z) \phi_l(x, y, z) dx\, dy\, dz$$

Substituting for $U(x, y, z)$ yields

$$\int_{\Omega} L\left(\sum_{m=1}^{nxyz} U_m \phi_m(x, y, z)\right) \phi_l(x, y, z) dx\, dy\, dz = \int_{\Omega} f(x, y, z) \phi_l(x, y, z) dx\, dy\, dz$$

Bringing the summation out of the integral yields

$$\sum_{m=1}^{nxyz} U_m \int_{\Omega} L\left(\phi_m(x, y, z)\right) \phi_l(x, y, z) dx\, dy\, dz = \int_{\Omega} f(x, y, z) \phi_l(x, y, z) dx\, dy\, dz$$

$L(\phi_m(x, y, z))$ means a substitution in $L(u(x, y, z))$ where $u(x, y, z)$ becomes $\phi_m(x, y, z)$, $ux(x, y, z)$ becomes $\phi_i^{'}(x, y, z)\, \phi_j(x, y, z)\, \phi_k(x, y, z)$, $uy(x, y, z)$ becomes $\phi_i(x, y, z)\, \phi_j^{'}(x, y, z)\, \phi_k(x, y, z)$, $uz(x, y, z)$ becomes $\phi_i(x, y, z)\, \phi_j(x, y, z)\, \phi_k^{'}(x, y, z)$, etc.

Writing the above in matrix form using the index $m = (i-1) \times ny \times nz + (j-1) \times nz + k$ for rows and index $l = (i-1) \times ny \times nz + (j-1) \times nz + k$ for columns yields

$$\begin{vmatrix} \int_\Omega L\left(\phi_1(x,y,z)\right)\phi_1(x,y,z)dx\,dy\,dz & \int_\Omega L\left(\phi_2(x,y,z)\right)\phi_1(x,y,z)dx\,dy\,dz & \dots & \int_\Omega L\left(\phi_{nxyz}(x,y,z)\right)\phi_1(x,y,z)dx \\ \int_\Omega L\left(\phi_1(x,y,z)\right)\phi_2(x,y,z)dx\,dy\,dz & \int_\Omega L\left(\phi_2(x,y,z)\right)\phi_2(x,y,z)dx\,dy\,dz & \dots & \int_\Omega L\left(\phi_{nxyz}(x,y,z)\right)\phi_2(x,y,z)dx \\ & & \dots & \\ \int_\Omega L\left(\phi_1(x,y,z)\right)\phi_{nxyz}(x,y,z)dx\,dy\,dz & \int_\Omega L\left(\phi_2(x,y,z)\right)\phi_{nxyz}(x,y,z)dx\,dy\,dz & \dots & \int_\Omega L\left(\phi_{nxyz}(x,y,z)\right)\phi_{nxyz}(x,y, \end{vmatrix}$$

$$\begin{vmatrix} U_1 \\ U_2 \\ \dots \\ U_{nxyz} \end{vmatrix} = \begin{vmatrix} \int_\Omega f(x,y,z)\phi_1(x,y,z)dx\,dy\,dz \\ \int_\Omega f(x,y,z)\phi_2(x,y,z)dx\,dy\,dz \\ \dots \\ \int_\Omega f(x,y,z)\phi_{nxyz}(x,y,z)dx\,dy\,dz \end{vmatrix}$$

Note that the above applies for the "internal" non boundary nodes.

Given Dirichlet boundary values v_1 at (x_1, y_1, z_1) and v_{nxyz} at (x_{nx}, y_{ny}, z_{nz}) the first and last rows of the above matrix equation would be:

$$\begin{vmatrix} 1\ 0\ \dots\ 0 \\ \dots \\ 0\ 0\ \dots\ 1 \end{vmatrix} \times \begin{vmatrix} U_1 \\ \dots \\ U_{nxyz} \end{vmatrix} = \begin{vmatrix} v_1 \\ \dots \\ v_{nxyz} \end{vmatrix}$$

The many boundary rows may be eliminated and a $(nx-2)(ny-2)(nz-2)$ system of equations are solved to find the U_m for $i = 2 \dots nx-1, j = 2 \dots ny-1, k = 2 \dots nz-1$

Note that, in general, numerical integration is required to compute the matrix elements and the right hand side vector elements.

Given a four dimensional linear partial differential equation with dependent variable u and independent variables x, y, z, t

$$L\left(u(x, y, z, t)\right) = f(x, y, z, t)$$

L is a general linear differential operator of some specific order. Samples for up to fourth order in up to four dimensions are shown below.

Find the approximate solution $U(x, y, z, t)$ numerically, that has boundary conditions chosen that make the problem of finding $u(x, y, z)$ well posed on the domain Ω for all $(x, y, z, t) \in \Omega$.

We denote the approximated solution $U(x_i, y_j, z_k, t_p)$ as U_m at node (x_i, y_j, z_k, t_p).

We take $i = 1 \ldots nx$ for specific nodes $x_1, x_2, \ldots, x_{nx}$ and $j = 1 \ldots ny$ for specific nodes $y_1, y_2, \ldots, y_{ny}$. $k = 1 \ldots nz$ for specific nodes $z_1, z_2, \ldots, z_{nz}$. $p = 1 \ldots nt$ for specific nodes $t_1, t_2, \ldots, t_{nt}$.

Thus, without loss of generality, m runs over all (x_i, y_j, z_k, t_p)
$m = (i-1) \times ny \times nz \times nt + (j-1) \times nz \times nt + (k-1) \times nt + p$ and
$nxyzt = nx \times ny \times nz \times nt$.

Let $U(x, y, z, t) = \sum_{m=1}^{nxyzt} U_m \phi_m(x, y, z, t)$

We will use

$$\phi_m(x, y, z, t) = \phi_i(x)\phi_j(y)\phi_k(z)\phi_p(t)$$

where $\phi_m(x, y, z, t)$ is the m_{th} polynomial in a set of four dimensional orthogonal polynomials such that:

$$\phi_m(x_i, y_j, z_k, t_p) = \begin{cases} 1 \ for \ i = m, j = m, k = m, p = m \\ 0 \ otherwise \end{cases}$$

We will use ϕ as a set of Lagrange orthogonal polynomials.

The Galerkin Method states:

$$\int_\Omega L\left(U(x, y, z)\right) \phi_l(x, y, z, t) dx\, dy\, dz\, dt = \int_\Omega f(x, y, z, t) \phi_l(x, y, z, t) dx\, dy\, dz\, dt$$

Substituting for $U(x, y, z, t)$ yields

$$\int_\Omega L\left(\sum_{m=1}^{nxyzt} U_m \phi_m(x, y, z, t)\right) \phi_l(x, y, z, t) dx\, dy\, dz\, dt = \int_\Omega f(x, y, z, t) \phi_l(x, y, z, t) dx\, dy\, dz\, dt$$

Bringing the summation out of the integral yields

$$\sum_{m=1}^{nxyzt} U_m \int_\Omega L\left(\phi_m(x, y, z, t)\right) \phi_l(x, y, z, t) dx\, dy\, dz\, dt = \int_\Omega f(x, y, z, t) \phi_l(x, y, z, t) dx\, dy\, dz\, dt$$

$L(\phi_m(x, y, z, t))$ means a substitution in $L(u(x, y, z, t))$ where $u(x, y, z, t)$ becomes $\phi_m(x, y, z, t)$, $ux(x, y, z, t)$ becomes $\phi_i'(x, y, z, t)\, \phi_j(x, y, z, t)\, \phi_k(x, y, z, t)\, \phi_p(x, y, z, t)$, $uy(x, y, z, t)$ becomes $\phi_i(x, y, z, t)\, \phi_j'(x, y, z,$ $uz(x, y, z, t)$ becomes $\phi_i(x, y, z, t)\, \phi_j(x, y, z, t)\, \phi_k'(x, y, z, t)\, \phi_p(x, y, z, t)$, $ut(x, y, z, t)$ becomes $\phi_i(x, y, z, t)\, \phi_j(x, y, z, t$ etc.

Writing the above in matrix form using the index $m = (i-1) \times ny \times nz \times nt + (j-1) \times nz \times nt + (k-1) \times nt + p$ for rows and index $l = (i-1) \times ny \times nz \times nt + (j-1) \times nz \times nt + (k-1) \times nt + p$ for columns yields

$$\left| \begin{array}{cccc} \int_\Omega L\left(\phi_1(x,y,z,t)\right)\phi_1(x,y,z,t)dx\,dy\,dz\,dt & \int_\Omega L\left(\phi_2(x,y,z,t)\right)\phi_1(x,y,z,t)dx\,dy\,dz\,dt & \ldots & \int_\Omega L\left(\phi_{nxyzt}(x,y,z, \right. \\ \int_\Omega L\left(\phi_1(x,y,z,t)\right)\phi_2(x,y,z,t)dx\,dy\,dz\,dt & \int_\Omega L\left(\phi_2(x,y,z,t)\right)\phi_2(x,y,z,t)dx\,dy\,dz\,dt & \ldots & \int_\Omega L\left(\phi_{nxyzt}(x,y,z, \right. \\ & & \ldots & \\ \int_\Omega L\left(\phi_1(x,y,z,t)\right)\phi_{nxyzt}(x,y,z,t)dx\,dy\,dz\,dt & \int_\Omega L\left(\phi_2(x,y,z,t)\right)\phi_{nxyzt}(x,y,z,t)dx\,dy\,dz\,dt & \ldots & \int_\Omega L\left(\phi_{nxyzt}(x,y, \right. \end{array} \right.$$

$$\left| \begin{array}{c} U_1 \\ U_2 \\ \ldots \\ U_{nxyzt} \end{array} \right| = \left| \begin{array}{c} \int_\Omega f(x,y,z,t)\phi_1(x,y,z,t)dx\,dy\,dz\,dt \\ \int_\Omega f(x,y,z,t)\phi_2(x,y,z,t)dx\,dy\,dz\,dt \\ \ldots \\ \int_\Omega f(x,y,z,t)\phi_{nxyzt}(x,y,z,t)dx\,dy\,dz\,dt \end{array} \right|$$

Note that the above applies for the "internal" non boundary nodes.

Given Dirichlet boundary values v_1 at (x_1, y_1, z_1, t_1) and v_{nxyzt} at $(x_{nx}, y_{ny}, z_{nz}, t_{nt})$ the first and last rows of the above matrix equation would be:

$$\left| \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & 1 \end{array} \right| \times \left| \begin{array}{c} U_1 \\ \ldots \\ U_{nxyzt} \end{array} \right| = \left| \begin{array}{c} v_1 \\ \ldots \\ v_{nxyzt} \end{array} \right|$$

The many boundary rows may be eliminated and a $(nx-2)(ny-2)(nz-2)(nt-2)$ system of equations are solved to find the U_m for $i = 2 \ldots nx-1, j = 2 \ldots ny-1, k = 2 \ldots nz-1, p = 2 \ldots nt-1$

Note that, in general, numerical integration is required to compute the matrix elements and the right hand side vector elements.

Some general formulations of linear partial differential equations from first order to fourth order with one to four dimensions. The functions $f_i()$ may be constants including zero. The linear differential operator L used above, may be any left hand side from these, or more general, equations.

$$f_2(x)\frac{\partial u(x)}{\partial x} + f_1(x)u(x) = f(x)$$

$$f_3(x)\frac{\partial^2 u(x)}{\partial x^2} + f_2(x)\frac{\partial u(x)}{\partial x} + f_1(x)u(x) = f(x)$$

$$f_4(x)\frac{\partial^3 u(x)}{\partial x^3} + f_3(x)\frac{\partial^2 u(x)}{\partial x^2} + f_2(x)\frac{\partial u(x)}{\partial x} + f_1(x)u(x) = f(x)$$

$$f_5(x)\frac{\partial^4 u(x)}{\partial x^4} + f_4(x)\frac{\partial^3 u(x)}{\partial x^3} + f_3(x)\frac{\partial^2 u(x)}{\partial x^2} + f_2(x)\frac{\partial u(x)}{\partial x} + f_1(x)u(x) = f(x)$$

$$f_3(x,y)\frac{\partial u(x,y)}{\partial x} + f_2(x,y)\frac{\partial u(x,y)}{\partial y} + f_1(x,y)u(x,y) = f(x,y)$$

$$f_6(x,y)\frac{\partial^2 u(x,y)}{\partial x^2} + f_5(x,y)\frac{\partial^2 u(x,y)}{\partial xy} + f_4(x,y)\frac{\partial^2 u(x,y)}{\partial y^2} +$$

$$f_3(x,y)\frac{\partial u(x,y)}{\partial x} + f_2(x,y)\frac{\partial u(x,y)}{\partial y} + f_1(x,y)u(x,y) = f(x,y)$$

$$f_{10}(x,y)\frac{\partial^3 u(x,y)}{\partial x^3} + f_9(x,y)\frac{\partial^3 u(x,y)}{\partial x^2 y} + f_8(x,y)\frac{\partial^3 u(x,y)}{\partial xy^2} +$$

$$f_7(x,y)\frac{\partial^3 u(x,y)}{\partial y^3} + f_6(x,y)\frac{\partial^2 u(x,y)}{\partial x^2} + f_5(x,y)\frac{\partial^2 u(x,y)}{\partial xy} +$$

$$f_4(x,y)\frac{\partial^2 u(x,y)}{\partial y^2} + f_3(x,y)\frac{\partial u(x,y)}{\partial x} + f_2(x,y)\frac{\partial u(x,y)}{\partial y} + f_1(x,y)u(x,y) = f(x,y)$$

$$f_4(x,y,z)\frac{\partial u(x,y,z)}{\partial x} + f_3(x,y,z)\frac{\partial u(x,y,z)}{\partial y} + f_2(x,y,z)\frac{\partial u(x,y,z)}{\partial z} +$$

$$f_1(x,y,z)u(x,y,z) = f(x,y,z)$$

$$f_{10}(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial x^2} + f_9(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial xy} + f_8(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial xz} +$$

$$f_7(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial y^2} + f_6(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial yz} + f_5(x,y,z)\frac{\partial^2 u(x,y,z)}{\partial z^2} +$$

$$f_4(x,y,z)\frac{\partial u(x,y,z)}{\partial x} + f_3(x,y,z)\frac{\partial u(x,y,z)}{\partial y} + f_2(x,y,z)\frac{\partial u(x,y,z)}{\partial z} +$$

$$f_1(x,y,z)u(x,y,z) = f(x,y,z)$$

$$f_5(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial x} + f_4(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial y} + f_3(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial z} +$$

$$f_2(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial t} + f_1(x,y,z,t)u(x,y,z,t) = f(x,y,z,t)$$

$$f_{14}(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + f_{13}(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial xy} + f_{12}(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial xz} +$$

$$f_{11}(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial xt} + f_{10}(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial y^2} + f_9(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial yz} +$$

$$f_8(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial yt} + f_7(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial z^2} + f_6(x,y,z,t)\frac{\partial^2 u(x,y,z,t)}{\partial zt} +$$

$$f_5(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial x} + f_4(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial y} + f_3(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial z} +$$

$$f_2(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial t} + f_1(x,y,z,t)u(x,y,z,t) = f(x,y,z,t)$$

$$f_{20}(x,y,z,t)\frac{\partial^4 u(x,y,z,t)}{\partial x^4} + f_{19}(x,y,z,t)\frac{\partial^4 u(x,y,z,t)}{\partial y^4} + f_{18}(x,y,z,t)\frac{\partial^4 u(x,y,z,t)}{\partial z^4} +$$

$$f_{17}(x,y,z,t)\frac{\partial^4 u(x,y,z,t)}{\partial t^4} + \ldots + f_5(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial x} + f_4(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial y} +$$

$$f_3(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial z} + f_2(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial t} + f_1(x,y,z,t)u(x,y,z,t) = f(x,y,z,t)$$

Lagrange polynomials and derivatives of various orders and dimensions

One dimension, using $\phi(x)$ for the Lagrange polynomial on the points $x_1, x_2, \ldots, x_n$, for a given j and x we can numerically compute

$$\phi_j(x) = \prod_{i=1, i\neq j}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Two dimensions, using $\phi_i(x)$ and $\phi_j(y)$ for the Lagrange polynomials on the points $x_1, x_2, \ldots, x_{nx}$,$y_1, y_2, \ldots, y_{ny}$, for a given i and x, j and y we can numerically compute

$$\phi_m(x, y) = \phi_i(x)\phi_j(y)$$

This is just the product of the evaluations of the two Lagrange polynomials. Note that the two polynomials may be of different degree and have different indices.

Similarly, we can numerically compute

$$\phi_m(x, y, z) = \phi_i(x)\phi_j(y)\phi_k(z)$$

$$\phi_m(x, y, z, t) = \phi_i(x)\phi_j(y)\phi_k(z)\phi_p(t)$$

Taking the derivative with respect to x gives

$$\phi_j^{'}(x) = \sum_{k=1, k\neq j}^{n} \frac{1}{x_j - x_i} \prod_{i=1, i\neq j}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Higher order derivatives may be defined in the same way.

For numerical computation:

The numerical integration is best performed by Gauss-Legendre quadrature, my experience with adaptive quadrature was not good.

The simultaneous equations may be solved by any numerically stable method.