STUDY PROBLEMS FOR EXAM 2
CMSC 203 FALL 2007
DISCRETE STRUCTURES
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1. Use the principle of mathematical induction to prove that

$$P(n): \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

for all integers $n \geq 1$.

Answer:

Proof (by weak induction):

Basis Step: $P(n)$ is true for $n = 1$, for:

$$\sum_{j=1}^{1} j^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

Inductive Hypothesis: Assume for a fixed but arbitrary integer $k \geq 1$ that $P(k)$ is true, i.e., that

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

[We start with the left hand side and transform it using the inductive hypothesis into the right hand side.]

$\sum_{j=1}^{k+1} j^2$	$=$	$\left(\sum_{j=1}^{k} j^2\right) + (k+1)^2$	**Reason:** Basic algebra
	$=$	$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$	**Reason:** Ind. Hypoth. &substitution
	$=$	$\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$	**Reason:** Basic algebra
	$=$	$\frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right]$	**Reason:** Basic algebra
	$=$	$\frac{(k+1)}{6}\left[2k^2+k+6k+6\right]$	**Reason:** Basic algebra
	$=$	$\frac{(k+1)}{6}\left(2k^2+7k+6\right)$	**Reason:** Basic algebra
	$=$	$\frac{(k+1)}{6}(k+2)(2k+3)$	**Reason:** Basic algebra
	$=$	$\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$	**Reason:** Basic algebra

Thus, we have used the inductive hypothesis to prove that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$$

Magic Wand Step: By the P.M.I.,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 1$$

□

Q.E.D.

2. Use the principle of mathematical induction to prove that

$$\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

3. Let d_1, d_2, d_3, … be the sequence defined by

$$d_j = d_{j-1} \cdot d_{j-2} \text{ for all integers } j \geq 3$$

and

$$d_1 = \frac{9}{10} \quad \text{and} \quad d_2 = \frac{10}{11}$$

Use math induction to prove that

$$P(n) : d_n \leq 1 \text{ for all integers } n \geq 1.$$

Proof (by strong induction):

Basis Step: Both $P(1)$ and $P(2)$ are true, for:

$$\begin{cases} d_1 = \frac{9}{10} \leq 1 & \textbf{Reason: } \text{Definition of } d_1 \\ d_2 = \frac{10}{11} \leq 1 & \textbf{Reason: } \text{Definition of } d_2 \end{cases}$$

Inductive Hypothesis: Assume for a fixed but arbitrary integer $k > 2$ that $P(\ell)$ is true for $1 \leq \ell < k$, i.e., that

$$d_\ell \leq 1 \text{ for } 1 \leq \ell < k$$

Inductive Step: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that

$$d_k \leq 1$$

$$\begin{array}{lll} & d_k = d_{k-1} \cdot d_{k-2} & \textbf{Reason: } \text{Definition of } d_k \\ \text{But} & d_{k-1} \leq 1 \text{ and } d_{k-2} \leq 1 & \textbf{Reason: } \text{Ind. Hypoth.} \\ \text{thus,} & d_k \leq 1 & \textbf{Reason: } \text{Basic algebra} \end{array}$$

Magic Wand Step: Hence, by. the P.M.I.,

$$d_n \leq 1 \quad \text{for} \quad n \geq 1$$

□

Q.E.D.

4. Let e_0, e_1, e_2, . . . be the sequence defined by

$$e_j = e_{j-1} + e_{j-2} + e_{j-3} \quad \text{for all integers } j \geq 3$$

and

$$e_0 = 1, \quad e_1 = 2, \quad e_2 = 3$$

Use math induction to prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

5. Simplify the followng product as much as possible

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right)$$

The answer is

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right) = \left(\prod_{j=1}^{n} 2\right) \cdot \left(\prod_{j=1}^{n} 4^j\right) = 2^n \cdot 4^{\sum_{j=1}^{n} j}$$

But $\sum_{j=1}^{n} j$ is the arithmetic series. Hence, $\sum_{j=1}^{n} j = n\,(n+1)\,/2$. Thus,

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right) = 2^n \cdot 4^{\sum_{j=1}^{n} j} = 2^n \cdot 4^{n(n+1)/2} = 2^n \cdot 2^{2[n(n+1)/2]} = 2^n \cdot 2^{n(n+1)} = 2^{n^2+2n}$$

6. Simplify the following sum as much as possible

$$\sum_{j=0}^{n}(2+4k)$$

The answer is

$$\sum_{j=0}^{n}(2+4k)=\sum_{j=0}^{n}2+\sum_{j=0}^{n}4k=2(n+1)+4\sum_{j=0}^{n}k=4\cdot\frac{n(n+1)}{2}=2n(n+1)$$

7. Transform the following product by making the change of variable $i=k+1$

$$\prod_{k=1}^{n}\frac{k}{k^2+4}$$

The answer is:

$$\prod_{k=1}^{n}\frac{k}{k^2+4}=\prod_{i=2}^{n+1}\frac{i-1}{(i-1)^2+4}=\prod_{i=2}^{n+1}\frac{i-1}{i^2-2i+5}$$

8. Simplify the following product as much as possible

$$\prod_{j=2}^{5}\frac{(j-1)\cdot j}{(j+1)\cdot(j+2)}$$

The answer is:

$$\prod_{j=2}^{5}\frac{(j-1)\cdot j}{(j+1)\cdot(j+2)}=\left(\frac{1\cdot 2}{3\cdot 4}\right)\cdot\left(\frac{2\cdot 3}{4\cdot 5}\right)\cdot\left(\frac{3\cdot 4}{5\cdot 6}\right)\cdot\left(\frac{4\cdot 5}{6\cdot 7}\right)$$

which simplifies to

$$\frac{1}{3\cdot 5\cdot 7}=\frac{1}{105}$$

9. Let R be the binary relation on $\{\ 0,\ 1,\ 2\ \}$ defined by

$$R=\{\quad(0,1),\quad(1,2),\quad(2,0)\quad\}$$

Compute:

a) $R^2=R\circ R$ **b)** $R^3=R\circ R\circ R$ **c)** $R^4=R\circ R\circ R\circ R$ **d)** The transive closure R^t of R

10. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $g:\mathbb{R}\longrightarrow\mathbb{R}$ be functions defined respectively by

$$f(x)=3x-5\qquad\text{and}\qquad g(x)=2-7x$$

then

a) $(f\circ g)(x)=\ ?$ **b)** $\left(f^{-1}\right)(x)=\ ?$

11. Let R denote the equivalence relation on the set $\mathbb{S}=\{n\in\mathbb{Z}\quad:\quad|n|\leq 5\}$ defined by

$$\forall m,n\in\mathbb{S}\ ,\ mRn\Longleftrightarrow 3\text{ exactly divides }m^2-n^2$$

List all the distinct equivalence classes of R. (List each equivalence class by writing down all the elements in that class.)

12. Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \frac{x+1}{x}$$

Is f injective? Please explain your answer.

13. Let $g : \mathbb{R} - \{-1\} \longrightarrow \mathbb{R}$ be the function defined by

$$g(x) = \frac{x-1}{x+1}$$

Is g surjective? Please explain your answer.

14. Let R_1 be the binary relation defined on $\mathbb{R}$ as follows:

$$\text{For all } x, y \in \mathbb{R},\ xR_1y \Longleftrightarrow xy \geq 0$$

Is R_1 reflexive? Is it symmetric? Is it transitive? Is it antisymmetric? Please explain your answers.

15. Let A be a nonempty set, and let $\mathcal{P}(A)$ be the power set of A. Define a relation R_2 on $\mathcal{P}(A)$ as follows:

$$\text{For all } X, Y \in \mathcal{P}(A),\ XR_2Y \Longleftrightarrow X \neq Y$$

Is R_2 reflexive? Is it symmetric? Is it transitive? Is it antisymmetric? Please explain your answers.

16. On the set $\{0, 1, 2, 3, 4\}$ consider the equivalence relation

$$R = \{\ (0,0),\ (0,4),\ (1,1),\ (1,3),\ (2,2),\ (3,1),\ (3,3),\ (4,0),\ (4,4)\ \}$$

List all the distinct equivalence classes of R

17. Let

$$f : A \longrightarrow B \quad \text{and} \quad g : B \longrightarrow A$$

be functions such that

$$g \circ f = id_A$$

where $id_A : A \longrightarrow A$ is the identity function on A. Prove that f is injective, i.e., one-to-one.

Hint: First give the definition of an injective function, and then use your definition to prove that f is injective.

18. Let f, g, h, k be functions from the set $S = \{1, 2, 3, 4\}$ to itself repectively defined by

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ 1 & \longmapsto & 2 \\ 2 & \longmapsto & 1 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{g} & S \\ 1 & \longmapsto & 2 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 4 \\ 4 & \longmapsto & 3 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{h} & S \\ 1 & \longmapsto & 4 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{k} & S \\ 1 & \longmapsto & 2 \\ 2 & \longmapsto & 1 \\ 3 & \longmapsto & 4 \\ 4 & \longmapsto & 3 \end{array}$$

a) Is f injective? Why?
b) Is f surjective? Why?
c) Is g onto? Why?

d) Is g one-to-one? Why?
e) Is h is one-to-one? Why?
f) Is k a bijection? Why?
g) Does the inverse function $k^{-1} : S \longrightarrow S$ exist? Why?
h) Is $f \circ k$ a bijection? Why?
i) Compute $(f \circ g \circ h \circ k)(2)$

Answer: Since

$$\begin{array}{ccccccccc} S & \xrightarrow{k} & S & \xrightarrow{h} & S & \xrightarrow{g} & S & \xrightarrow{f} & S \\ 2 & \longmapsto & 1 & \longmapsto & 4 & \longmapsto & 3 & \longmapsto & 3 \end{array}$$

it follows that $(f \circ g \circ h \circ k)(2) = 3$

j) Compute $(f \circ g)(1)$. Please show how you computed your answer.
k) Compute $(g \circ f)(1)$. Please show how you computed your answer.

19. Find the unique integer x in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that

$$7x = 1 \pmod{11}$$

20. Find the unique integer x in $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ such that

$$(723)^2 \cdot (16756) = x \pmod 9$$