



## Quantum Knots & Mosaics

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L-O-O-P

Throughout this talk:

"Knot" means either a knot or a link

This talk is based on the paper:

Lomonaco and Kauffman, *Quantum Knots and Mosaics*, *Journal of Quantum Information Processing*, vol. 7, Nos. 2-3, (2008), 85-115. An earlier version can be found at: <http://arxiv.org/abs/0805.0339>

This talk was motivated by:

Lomonaco, Samuel J., Jr., *The modern legacies of Thomson's atomic vortex theory in classical electrodynamics*, *AMS PSAPM/51*, Providence, RI (1996), 145 - 166.

Kauffman and Lomonaco, *Quantum Knots*, *SPIE Proc. on Quantum Information & Computation II* (ed. by Donkor, Pirich, & Brandt), (2004), 5436-30, 268-284.  
<http://xxx.lanl.gov/abs/quant-ph/0403228>

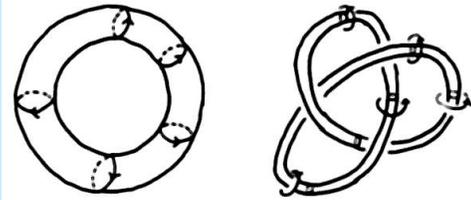
This talk was also motivated by:

**Kitaev, Alexei Yu, Fault-tolerant quantum computation by anyons, <http://arxiv.org/abs/quant-ph/9707021>**

**Rasetti, Mario, and Tullio Regge, Vortices in He II, current algebras and quantum knots, Physica 80 A, North-Holland, (1975), 217-2333.**

### What Motivated This Talk ?

#### Classical Vortices in Plasmas



**Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.**

### Knots Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots in quantum physics:  
**Knotted vortices**

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

**Reason for current intense interest:  
A Natural Topological Obstruction to Decoherence**

### Objectives

- We seek to create a quantum system that simulates a closed knotted physical piece of rope.
- We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
- We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

### Rules of the Game

Find a mathematical definition of a quantum knot that is

- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

### Aspirations

We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

**Overview**

Part 0. Quick Overview of Knot Theory

Part 1. Mosaic Knots

We reduce tame knot theory to a formal system of string manipulation rules, i.e., string rewriting systems.

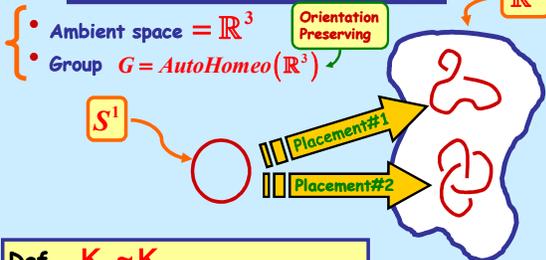
Part 2. Quantum Knots

We then use mosaic knots to build a physically implementable definition of quantum knots.

**Quick Overview to Knot Theory**

 Skip to mosaic knots

**Placement Problem: Knot Theory**



**Def.**  $K_1 \sim K_2$  if  $g \in G$  s.t.  $gK_1 = K_2$

**Problem.** When are two placements the same?  
 $K_1 \sim K_2$  ?

**What is a knot invariant ?**

**Def.** A knot invariant  $I$  is a map

$$I : \text{Knots} \rightarrow \text{Mathematical Domain}$$

that takes each knot  $K$  to a mathematical object  $I(K)$  such that

$$K_1 \sim K_2 \Rightarrow I(K_1) = I(K_2)$$

Consequently,

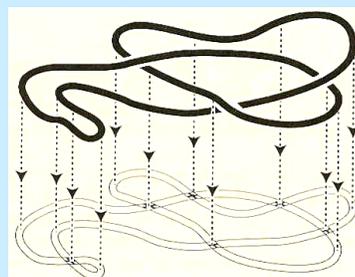
$$I(K_1) \neq I(K_2) \Rightarrow K_1 \neq K_2$$

**The Jones polynomial is a knot invariant.**

**Knot Diagrams:**

A fundamental tool in knot theory.

**Knot Projections**



**Knot Diagram**

- Planar four valent graph with
- Labeled vertices

**Question:** If we locally move the rope, what does its shadow (knot diagram) do ???

**Planar Isotopy Moves**

In this case, we have not changed the topological type of the knot diagram

This is a planar isotopy move denoted by R0

**Planar Isotopy Moves**

R0 ↔

This is a local move !

It does not change the topological type of the knot diagram.

**Reidemeister Moves**

R1

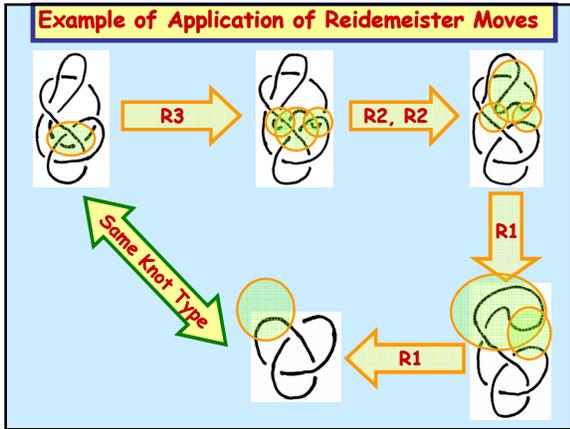
R2

R3

These are local moves that change the topological type of the knot diagram !

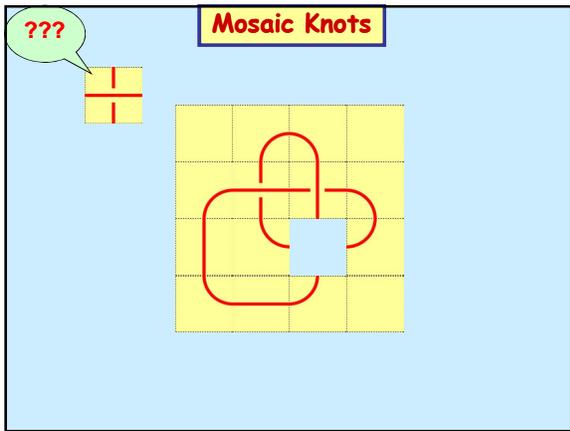
**When do two Knot diagrams represent the same or different knots ?**

**Theorem (Reidemeister).** Two knots diagrams represent the same knot iff one can be transformed into the other by a finite sequence of Reidemeister moves (and planar isotopy rules).



Part 1

# Mosaic Knots



**Mosaic Tiles**

Let  $T^{(u)}$  denote the following set of 11 symbols, called **mosaic (unoriented) tiles**:

Please note that, up to rotation, there are exactly 5 tiles

**Definition of an n-Mosaic**

An **n-mosaic** is an  $n \times n$  matrix of tiles, with rows and columns indexed  $0, 1, \dots, n-1$

An example of a 4-mosaic

**Tile Connection Points**

A **connection point** of a tile is a midpoint of a tile edge which is also the endpoint of a curve drawn on the tile. For example,

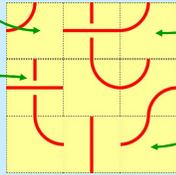
0 Connection Points      2 Connection Points      4 Connection Points

### Contiguous Tiles

Two tiles in a mosaic are said to be **contiguous** if they lie immediately next to each other in either the **same row** or the **same column**.

Contiguous

Not Contiguous

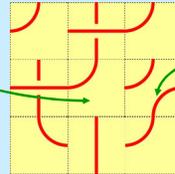


### Suitably Connected Tiles

A tile in a mosaic is said to be **Suitably Connected** if all its connection points touch the connection points of contiguous tiles. For example,

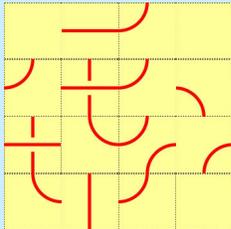
Suitably Connected

Not Suitably Connected

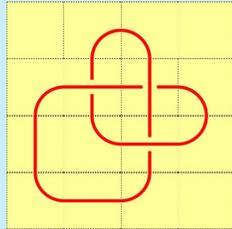


### Knot Mosaics

A **knot mosaic** is a mosaic with all tiles suitably connected. For example,

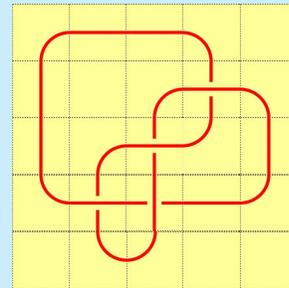


Non-Knot 4-Mosaic

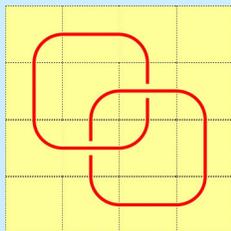


Knot 4-Mosaic

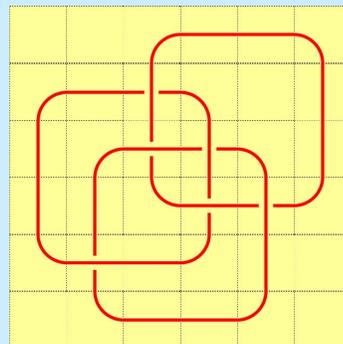
### Figure Eight Knot 5-Mosaic



### Hopf Link 4-Mosaic



### Borromean Rings 6-Mosaic



**Notation**

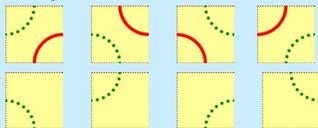
$M^{(n)}$  = Set of  $n$ -mosaics

$K^{(n)}$  = Subset of knot  $n$ -mosaics

**Planar Isotopy Moves**

**Non-Deterministic Tiles**

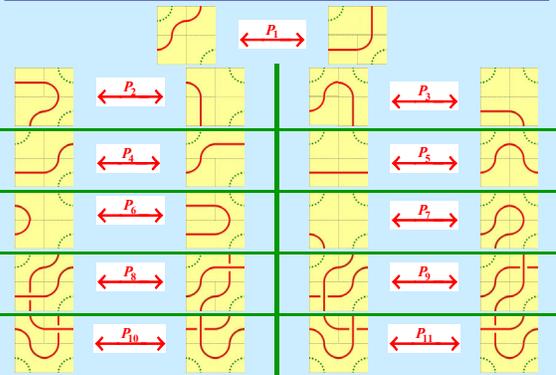
We use the following tile symbols to denote one of two possible tiles:



For example, the tile  denotes either



**11 Planar Isotopy (PI) Moves on Mosaics**

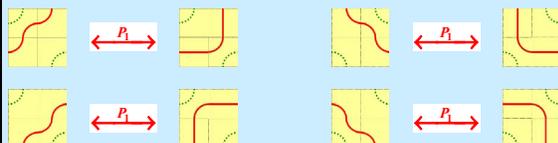


**Planar Isotopy (PI) Moves on Mosaics**

It is understood that each of the above moves depicts all moves obtained by rotating the  $2 \times 2$  sub-mosaics by 0, 90, 180, or 270 degrees.

For example,   $\xleftrightarrow{P_1}$  

represents each of the following 4 moves:



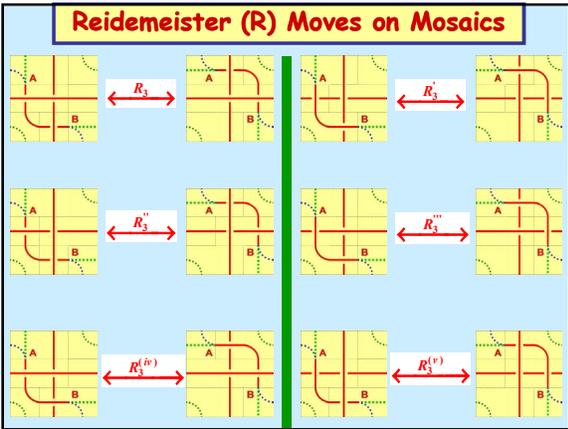
**Terminology: k-Submosaic Moves**

**Def.** A  $k$ -submosaic move on a mosaic  $M$  is a mosaic move that replaces one  $k$ -submosaic in  $M$  by another  $k$ -submosaic.

All of the PI moves are examples of 2-submosaic moves. I.e., each PI move replaces a 2-submosaic by another 2-submosaic

For example,   $\xleftrightarrow{P_1}$  





**Reidemeister (R) Moves on Mosaics**

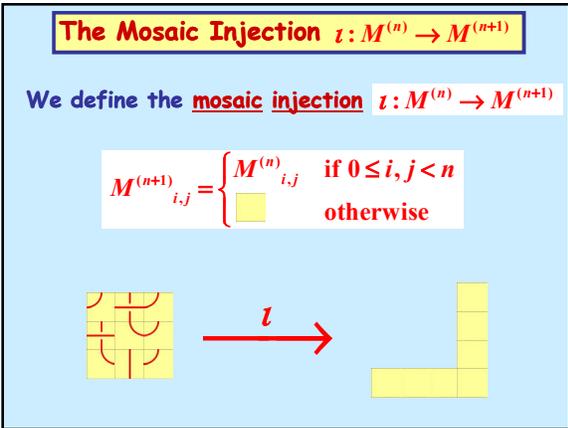
Just like each PI move, each R move is a permutation of the set of all knot  $n$ -mosaics  $K^{(n)}$

In fact, each R move, as a permutation, is a product of disjoint transpositions.

**The Ambient Group  $A(n)$**

We define the ambient isotopy group  $A(n)$  as the subgroup of the group of all permutations of the set  $K^{(n)}$  generated by the all PI moves and all Reidemeister moves.

**Knot Type**



**Mosaic Knot Type**

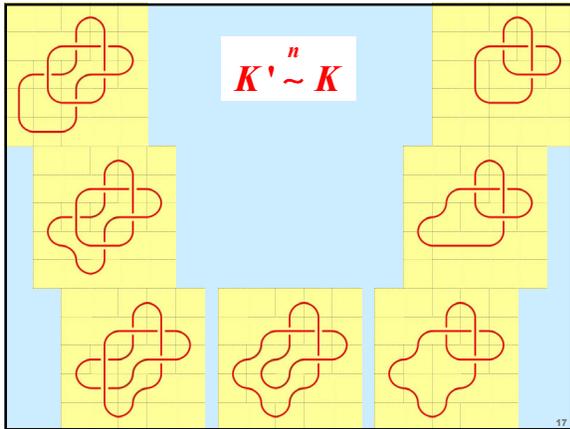
**Def.** Two  $n$ -mosaics  $M$  and  $M'$  are of the same knot  $n$ -type, written

$$M \stackrel{n}{\sim} M'$$

provided there exists an element of the ambient group  $A(n)$  that transforms  $M$  into  $M'$ .

Two  $n$ -mosaics  $M$  and  $M'$  are of the same knot type if there exists a non-negative integer  $k$  such that

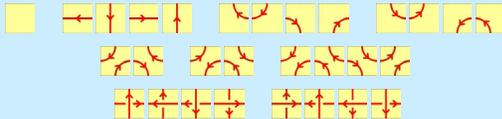
$$\iota^k M \stackrel{n+k}{\sim} \iota^k M'$$



# Oriented Mosaics

## Oriented Mosaics and Oriented Knot Type

In like manner, we can use the following oriented tiles to construct **oriented mosaics**, **oriented mosaic knots**, and **oriented knot type**



There are **29** oriented tiles, and **9** tiles up to rotation. Rotationally equivalent tiles have been grouped together.

Part 2

# Quantum Knots & Quantum Knot Systems

## The Hilbert Space $\mathcal{M}^{(n)}$ of n-mosaics

Let  $\mathcal{H}$  be the **11** dimensional Hilbert space with orthonormal basis labeled by the tiles



We define the **Hilbert space  $\mathcal{M}^{(n)}$  of n-mosaics** as

$$\mathcal{M}^{(n)} = \bigotimes_{k=0}^{n^2-1} \mathcal{H}$$

This is the Hilbert space with induced orthonormal basis

$$\left\{ \bigotimes_{k=0}^{n^2-1} |T_{\ell(k)}\rangle : 0 \leq \ell(k) < 11 \right\}$$

## The Hilbert Space $\mathcal{M}^{(n)}$ of n-mosaics

We identify each basis ket  $\bigotimes_{k=0}^{n^2-1} |T_{\ell(k)}\rangle$  with a ket  $|M\rangle$  labeled by an **n-mosaic  $M$**  using **row major order**.

For example, in the **3-mosaic** Hilbert space  $\mathcal{M}^{(3)}$ , the basis ket

$$|T_2\rangle \otimes |T_5\rangle \otimes |T_4\rangle \otimes |T_9\rangle \otimes |T_2\rangle \otimes |T_1\rangle \otimes |T_5\rangle \otimes |T_8\rangle \otimes |T_3\rangle$$

is identified with the **3-mosaic** labeled ket

$$\left| \begin{array}{ccc} T_2 & T_5 & T_4 \\ T_9 & T_2 & T_1 \\ T_5 & T_8 & T_3 \end{array} \right\rangle$$

**Identification via Row Major Order**

Let  $\mathcal{H}$  be the 11 dimensional Hilbert space with orthonormal basis labeled by the tiles



$T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad T_8 \quad T_9 \quad T_{10}$

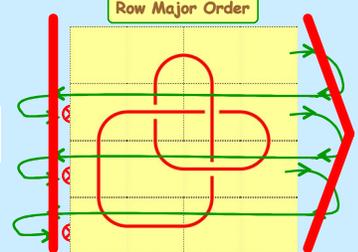
Construct Mosaic Space

$\mathcal{H}^{\otimes n^2} =$

$\bigotimes_{0 \leq i, j < n} |T_{k(i,j)}\rangle =$

Select Basis Element

Row Major Order



**The Hilbert Space  $\mathcal{K}^{(n)}$  of Quantum Knots**

The Hilbert space  $\mathcal{K}^{(n)}$  of quantum knots is defined as the sub-Hilbert space of  $\mathcal{M}^{(n)}$  spanned by all orthonormal basis elements labeled by knot  $n$ -mosaics.

**Quantum Knots**

We define the Hilbert space  $\mathcal{M}^{(n)}$  of  $n$ -mosaics as

$\mathcal{M}^{(n)} = \bigotimes_{k=0}^{n^2-1} \mathcal{H}$

This is the Hilbert space with induced orthonormal basis

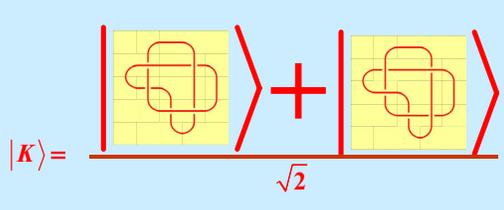
$\{ \bigotimes_{k=0}^{n^2-1} |T_{\ell(k)}\rangle : 0 \leq \ell < n^2 \}$

We identify each basis element  $\bigotimes_{k=0}^{n^2-1} |T_{\ell(k)}\rangle$  with the mosaic labeled ket  $|M\rangle$  via the bijection

$T_\ell \leftrightarrow M_{i,j}$  Row major order

where  $\begin{cases} i = \lfloor \ell/n \rfloor \\ j = \ell - n \lfloor \ell/n \rfloor \end{cases}$  and  $\ell = ni + j$

**An Example of a Quantum Knot**



$|K\rangle = \frac{1}{\sqrt{2}} (|M_1\rangle + |M_2\rangle)$

**The Ambient Group  $A(n)$  as a Unitary Group**

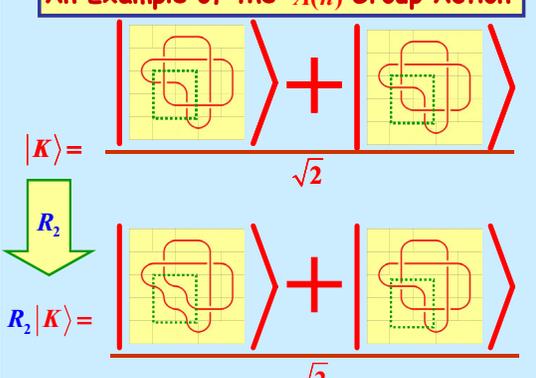
We identify each element  $g \in A(n)$  with the linear transformation defined by

$\mathcal{K}^{(n)} \rightarrow \mathcal{K}^{(n)}$   
 $|K\rangle \mapsto |gK\rangle$

Since each element  $g \in A(n)$  is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group  $A(n)$  becomes a discrete group of unitary transfs on the Hilbert space  $\mathcal{K}^{(n)}$ .

**An Example of the  $A(n)$  Group Action**



$R_2 |K\rangle = \frac{1}{\sqrt{2}} (|M_2\rangle + |M_1\rangle)$

### The Quantum Knot System $(\mathcal{K}^{(n)}, A(n))$

**Def.** A **quantum knot system**  $(\mathcal{K}^{(n)}, A(n))$  is a quantum system having  $\mathcal{K}^{(n)}$  as its state space, and having the Ambient group  $A(n)$  as its set of accessible unitary transformations.

The states of quantum system  $(\mathcal{K}^{(n)}, A(n))$  are **quantum knots**. The elements of the ambient group  $A(n)$  are **quantum moves**.

$$(\mathcal{K}^{(1)}, A(1)) \xrightarrow{i} \dots \xrightarrow{i} (\mathcal{K}^{(n)}, A(n)) \xrightarrow{i} (\mathcal{K}^{(n+1)}, A(n+1)) \xrightarrow{i} \dots$$

Physically Implementable

Physically Implementable

Physically Implementable

### The Quantum Knot System $(\mathcal{K}^{(n)}, A(n))$

$$(\mathcal{K}^{(1)}, A(1)) \xrightarrow{i} \dots \xrightarrow{i} (\mathcal{K}^{(n)}, A(n)) \xrightarrow{i} (\mathcal{K}^{(n+1)}, A(n+1)) \xrightarrow{i} \dots$$

Physically Implementable

Physically Implementable

Physically Implementable

Choosing an integer  $n$  is analogous to choosing a length of rope. The longer the rope, the more knots that can be tied.

The parameters of the ambient group  $A(n)$  are the "knobs" one turns to spacially manipulate the quantum knot.

### Quantum Knot Type

**Def.** Two quantum knots  $|K_1\rangle$  and  $|K_2\rangle$  are of the **same knot n-type**, written

$$|K_1\rangle \sim_n |K_2\rangle,$$

provided there is an element  $g \in A(n)$  s.t.

$$g|K_1\rangle = |K_2\rangle$$

They are of the **same knot type**, written

$$|K_1\rangle \sim |K_2\rangle,$$

provided there is an integer  $m \geq 0$  such that

$$i^m |K_1\rangle \sim_{n+m} i^m |K_2\rangle$$

### Two Quantum Knots of the Same Knot Type

$$|K\rangle = \frac{|\text{Knot 1}\rangle + |\text{Knot 2}\rangle}{\sqrt{2}}$$



$$R_2 |K\rangle = \frac{|\text{Knot 1}\rangle + |\text{Knot 2}\rangle}{\sqrt{2}}$$

### Two Quantum Knots NOT of the Same Knot Type

$$|K_1\rangle = |\text{Square Knot}\rangle$$

$$|K_2\rangle = \frac{|\text{Square Knot with +}\rangle + |\text{Square Knot with -}\rangle}{\sqrt{2}}$$

**Hamiltonians  
of the  
Generators  
of the  
Ambient Group**

### Hamiltonians for $A(n)$

Each generator  $g \in A(n)$  is the product of disjoint transpositions, i.e.,

$$g = (K_{\alpha_1}, K_{\beta_1}) (K_{\alpha_2}, K_{\beta_2}) \cdots (K_{\alpha_\ell}, K_{\beta_\ell})$$

Choose a permutation  $\eta$  so that

$$\eta^{-1} g \eta = (K_1, K_2) (K_3, K_4) \cdots (K_{\ell-1}, K_\ell)$$

Hence,

$$\eta^{-1} g \eta = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_1 & & & \\ & & \ddots & & \\ & & & \sigma_1 & \\ & & & & I_{n-2\ell} \end{pmatrix}, \text{ where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

### Hamiltonians for $A(n)$

Also, let  $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , and note that

$$\ln(\sigma_1) = \frac{i\pi}{2} (2s+1) (\sigma_0 - \sigma_1), \quad s \in \mathbb{Z}$$

For simplicity, we always choose the branch  $s=0$ .

$$H_g = -i\eta \ln(\eta^{-1} g \eta) \eta^{-1}$$

$$= \frac{\pi}{2} \eta \begin{pmatrix} I_\ell \otimes (\sigma_0 - \sigma_1) & 0 \\ 0 & 0_{(n-2\ell) \times (n-2\ell)} \end{pmatrix} \eta^{-1}$$

 Log of a matrix

### The Log of a Unitary Matrix

Let  $U$  be an arbitrary finite  $r \times r$  unitary matrix.

Then eigenvalues of  $U$  all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix  $W$  which diagonalizes  $U$ , i.e., there exists a unitary matrix  $W$  such that

$$WUW^{-1} = \Delta(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_r})$$

where  $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_r}$  are the eigenvalues of  $U$ .

### The Log of a Unitary Matrix

Then

$$\ln(U) = W^{-1} \Delta(\ln(e^{i\theta_1}), \ln(e^{i\theta_2}), \dots, \ln(e^{i\theta_r})) W$$

Since  $\ln(e^{i\theta_j}) = i\theta_j + 2\pi i n_j$ , where  $n_j \in \mathbb{Z}$  is an arbitrary integer, we have

$$\ln(U) = iW^{-1} \Delta(\theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \dots, \theta_r + 2\pi n_r) W$$

where  $n_1, n_2, \dots, n_r \in \mathbb{Z}$

### The Log of a Unitary Matrix

Since  $e^A = \sum_{m=0}^{\infty} A^m / (m!)$ , we have

$$\begin{aligned} e^{\ln(U)} &= e^{W^{-1} \Delta(\ln i\theta_1, \dots, \ln i\theta_r) W} \\ &= W^{-1} e^{\Delta(\ln i\theta_1, \dots, \ln i\theta_r)} W \\ &= W^{-1} \Delta(e^{\ln i\theta_1}, \dots, e^{\ln i\theta_r}) W \\ &= W^{-1} \Delta(e^{i\theta_1 + 2\pi i n_1}, \dots, e^{i\theta_r + 2\pi i n_r}) W \\ &= W^{-1} \Delta(e^{i\theta_1}, \dots, e^{i\theta_r}) W = U \end{aligned}$$

 Back

### Hamiltonians for $A(n)$

Using the Hamiltonian for the Reidemeister

2 move  $g = \begin{matrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{matrix} \xleftrightarrow{(2,1)} \begin{matrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{matrix}$

and the initial state 

we have that the solution to Schrodinger's equation for time  $t$  is

$$e^{\left( \frac{i\pi t}{2\hbar} \right)} \left( \cos\left( \frac{\pi t}{2\hbar} \right) \left| \begin{matrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{matrix} \right\rangle - i \sin\left( \frac{\pi t}{2\hbar} \right) \left| \begin{matrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{matrix} \right\rangle \right)$$

**Some  
Miscellaneous  
Unitary  
Transformations  
Not in  
 $A(n)$**

**Misc. Transformations**

The crossing tunneling transformation

$$\tau_{ij} = \begin{array}{|c|} \hline (i,j) \\ \hline \begin{array}{|c|} \hline \leftrightarrow \\ \hline \end{array} \\ \hline \end{array}$$

The mirror image transformation

$$\mu = \prod_{i,j=0}^{n-1} \left( \begin{array}{|c|} \hline (i,j) \\ \hline \begin{array}{|c|} \hline \leftrightarrow \\ \hline \end{array} \\ \hline \end{array} \right)$$

**Misc. Transformations**

The hyperbolic transformation

$$\eta_{ij} = \begin{array}{|c|} \hline (i,j) \\ \hline \begin{array}{|c|} \hline \leftrightarrow \\ \hline \end{array} \\ \hline \end{array}$$

The elliptic transformation

$$\varepsilon_{ij} = \begin{array}{|c|} \hline (i,j) \\ \hline \begin{array}{|c|} \hline \leftrightarrow \\ \hline \end{array} \\ \hline \end{array}$$

**Observables  
which are  
Quantum Knot  
Invariants**

**Observable Q. Knot Invariants**

**Question.** What do we mean by a physically observable knot invariant ?

Let  $(\mathcal{K}^{(n)}, A(n))$  be a quantum knot system. Then a quantum observable  $\Omega$  is a Hermitian operator on the Hilbert space  $\mathcal{K}^{(n)}$ .

**Observable Q. Knot Invariants**

**Question.** But which observables  $\Omega$  are actually knot invariants ?

**Def.** An observable  $\Omega$  is an invariant of quantum knots provided  $U\Omega U^{-1} = \Omega$  for all  $U \in A(n)$

### Observable Q. Knot Invariants

**Question.** But how do we find quantum knot invariant observables ?

**Theorem.** Let  $(\mathcal{K}^{(n)}, A(n))$  be a quantum knot system, and let

$$\mathcal{K}^{(n)} = \bigoplus_{\ell} W_{\ell}$$

be a decomposition of the representation  $A(n) \times \mathcal{K}^{(n)} \rightarrow \mathcal{K}^{(n)}$  into irreducible representations .

Then, for each  $\ell$  , the projection operator  $P_{\ell}$  for the subspace  $W_{\ell}$  is a quantum knot observable.

### Observable Q. Knot Invariants

**Theorem.** Let  $(\mathcal{K}^{(n)}, A(n))$  be a quantum knot system, and let  $\Omega$  be an observable on  $\mathcal{K}^{(n)}$ . Let  $St(\Omega)$  be the stabilizer subgroup for  $\Omega$  , i.e.,

$$St(\Omega) = \{ U \in A(n) : U\Omega U^{-1} = \Omega \}$$

Then the observable

$$\sum_{U \in A(n)/St(\Omega)} U\Omega U^{-1}$$

is a quantum knot invariant, where the above sum is over a complete set of coset representatives of  $St(\Omega)$  in  $A(n)$  .

### Observable Q. Knot Invariants

The following is an example of a quantum knot invariant observable:

$$\Omega = \left| \begin{array}{c} \text{Knot} \\ \text{Diagram} \end{array} \right\rangle \left\langle \begin{array}{c} \text{Knot} \\ \text{Diagram} \end{array} \right| + \left| \begin{array}{c} \text{Knot} \\ \text{Diagram} \end{array} \right\rangle \left\langle \begin{array}{c} \text{Knot} \\ \text{Diagram} \end{array} \right|$$

## Future Directions

&

## Open Questions

### Future Directions & Open Questions

- What is the structure of the ambient group  $A(n)$  and its direct limit  $A = \varinjlim A(n)$  ?  
Can one find a presentation of  $\vec{A}$  this group ?  
Is  $A(n)$  a Coxeter group?
- Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another ?  
If so, how ?

### Future Directions & Open Questions

- How does one find a quantum observable for the Jones polynomial ? This would be a family of observables parameterized by points on the circle in the complex plane. Does this approach lead to an algorithmic improvement to the quantum algorithm created by Aharonov, Jones, and Landau ?
- How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants ?

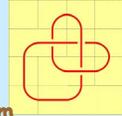
### Future Directions & Open Questions

- What is gained by extending the definition of quantum knot observables to POVMs ?
- What is gained by extending the definition of quantum knot observables to mixed ensembles ?

### Future Directions & Open Questions

**Def.** We define the **mosaic number** of a knot  $k$  as the smallest integer  $n$  for which  $k$  is representable as a knot  $n$ -mosaic.

- The mosaic number of the trefoil is **4**. In general, how does one compute the mosaic number of a knot? How does one find a quantum observable for the mosaic number?



- Is the mosaic number related to the crossing number of a knot?

### Future Directions & Open Questions

**Quantum Knot Tomography:** Given many copies of the same quantum knot, find the most efficient set of measurements that will determine the quantum knot to a chosen tolerance  $\epsilon > 0$ .

**Quantum Braids:** Use mosaics to define quantum braids. How are such quantum braids related to the work of Freedman, Kitaev, et al on anyons and topological quantum computing?

### Future Directions & Open Questions

- Can quantum knot systems be used to model and predict the behavior of
  - Quantum vortices in supercooled helium 2 ?
  - Quantum vortices in the Bose-Einstein Condensate
  - Fractional charge quantification that is manifest in the fractional quantum Hall effect

## UMBC Quantum Knots Research Lab

We at UMBC are very proud of our new state of the art Quantum Knots Research Laboratory.

### Quantum Knots Research Lab

