

EE 610/Spring 07/Project 1: Z transform and digital filters

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Abstract

In class we will study principles of filter design. Some of these filter design principles are easy (conceptually and computationally) and others are not. This project considers a simple method to create digital filters by starting with an analog design. The Bilinear transform is described in Oppenheim and Shafer, pages 450–458.

Students will first do some derivations followed by actual numerical calculations for filter design in MATLAB. They will hand in both their derivations and program that does the calculations. Upto fifty percent of the points for the project will be given for derivations and upto fifty percent for the MATLAB programs. The maximum a student can get is one hundred points.

Keywords: Bilinear transform, analog filters, low-pass filter design, MATLAB.

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1 The problem

Analog filters are described by their transfer function as

$$H(s) = \frac{A_0 + A_1s + A_2s^2 + \dots + A_ns^n}{B_0 + B_1s + B_2s^2 + \dots + B_ns^n}. \quad (1)$$

For ease of representation, the coefficients in the numerator and denominator are written in vector notation as

$$\begin{aligned} \mathbf{a}_a &= [A_0 \ A_1 \ A_2 \ \dots \ A_n] \\ \mathbf{b}_a &= [B_0 \ B_1 \ B_2 \ \dots \ B_n]. \end{aligned} \quad (2)$$

The subscript a indicates we are referring to “analog” and A_i, B_i are real coefficients.

On the other hand, digital filters are represented as

$$H(z) = \frac{E_0 + E_1z^{-1} + E_2z^{-2} + \dots + E_nz^{-n}}{F_0 + F_1z^{-1} + F_2z^{-2} + \dots + F_nz^{-n}}. \quad (3)$$

Thus we have

$$\begin{aligned} \mathbf{e}_d &= [E_0 \ E_1 \ E_2 \ \dots \ E_n] \\ \mathbf{f}_d &= [F_0 \ F_1 \ F_2 \ \dots \ F_n]. \end{aligned} \quad (4)$$

The subscript d indicates we are referring to “digital.”

The bilinear transform is written as

$$s = c \frac{z - 1}{z + 1}, \quad (5)$$

where c is a constant. If we substitute equation (5) in equation (1) we get a matrix equation that enables us to pass from \mathbf{a}_a and \mathbf{b}_a to \mathbf{e}_d and \mathbf{f}_d . This is written as:

$$\begin{aligned} \mathbf{e}_d &= \mathbf{P}_n \mathbf{a}_a^* \\ \mathbf{f}_d &= \mathbf{P}_n \mathbf{b}_a^* \end{aligned} \quad (6)$$

This equation says that we can go from analog to digital purely by simple matrix manipulations. Note that we are using \mathbf{a}_a^* and \mathbf{b}_a^* instead of \mathbf{a}_a and \mathbf{b}_a respectively. However there is a particularly simple relationship between \mathbf{a}_a^* and \mathbf{a}_a etc. Part of this project involves the identification of the mathematical structure of \mathbf{P}_n and the simple connection between \mathbf{a}_a^* and \mathbf{a}_a etc.

Start with $n = 2$ and see if you can get \mathbf{P}_n . Perform the same calculation for $n = 3$. This should give you a pattern to follow. Prove the higher values of n by induction. This technique is something I expect every person in my class to know. For a refresher go to http://en.wikipedia.org/wiki/Mathematical_induction.

2 An example

In order to help students check if their numbers are correct I assumed $c = 1$ and that

$$H(s) = \frac{s^2 + 5.153}{0.929s^3 + 2.781s^2 + 4.344s + 5.153}.$$

After working through the mathematics I obtained \mathbf{e}_d and \mathbf{f}_d . Rather than give you the numbers I instead plotted the amplitude and phase spectra for the obtained digital filter given in Figures 1 and 2. Check the numbers you obtain by using the appropriate MATLAB DSP function to help you create the amplitude and phase spectra.

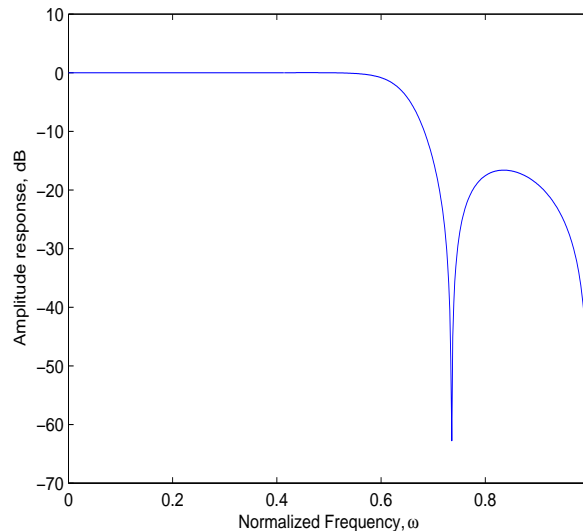


Figure 1: Gain of the digital filter. Students should note shape of response.

3 What to do?

This project has two parts. The first part deals with performing derivations in order to understand the basic theory while the latter one deals with programming in MATLAB.

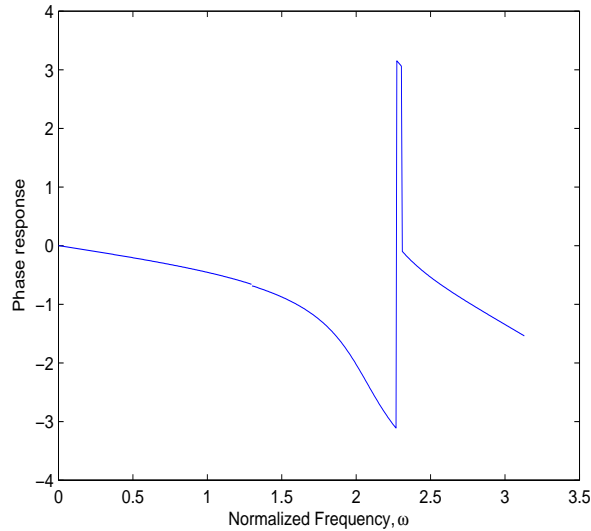


Figure 2: Phase response of the digital filter. Students should note non-linearities.

Derivations - 50%.

1. Obtain the relationship between \mathbf{a}_a^* and \mathbf{a}_a and do the same for \mathbf{b}_a^* and \mathbf{b}_a .
2. Provide \mathbf{P}_n for $n = 2, 3$, i.e., write the matrix explicitly.
3. Obtain the general form of \mathbf{P}_n in the pair of equations given by (6). This should be formulae that for example defines each term in the matrix. Use mathematical induction.
4. **Due date:** On or before 11:59pm on the 9 March 2007. Please drop off your calculations to Haleh Safavi in her office or scan and send to her by email.

MATLAB - 50%

- Write a MATLAB function *fnction* $[e, f, P] = \text{project1}(a, b, c)$ that solves the problem for general n . Your program will take the following inputs produce the following outputs
 1. **inputs.** Coefficient arrays for the analog filters \mathbf{a}_a and \mathbf{b}_a (from which \mathbf{a}_a^* and \mathbf{b}_a^* can be calculated); and c the constant in the transform.
 2. **outputs.** Coefficient arrays for the digital filters \mathbf{e}_d and \mathbf{f}_d . Also return the matrix \mathbf{P}_n . In addition your program will graph the amplitude (in dB) and phase responses against normalized frequencies.
- **Due date:** 11:59pm on 19 March 2006 by email only to Haleh Safavi.