

EE 610/Spring 06/Project 3: Matrix theory based filter design

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Abstract

In class we will do principles of filter design. Some of these filter design principles are easy (conceptually and computationally) and others are not. This project considers a deceptively simple method to design filters involving simple matrix algebra and a powerful minimization theorem from matrix calculus. Once the basic integrals are mastered, the deceptively simple method consists of obtaining the eigenvalues and eigenvectors of a particular symmetric matrix from which we obtain the particular filter.

Students will first do some derivations followed by actual numerical calculations for filter design in MATLAB. They will hand in both their derivations and program that does the calculations. Upto fifty percent of the points for the project will be given for derivations and upto fifty percent for the MATLAB programs. The maximum a student can get is one hundred points.

Keywords: Rayleigh's theorem, quadratic forms, symmetric matrices, FIR filter design, MATLAB.

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1 The problem

The frequency response of an FIR digital filter with real coefficients $a[n], n \in \{0, 1, 2, \dots, N-1\}$ is

$$\begin{aligned} H[\omega] &= \sum_{n=0}^{N-1} a[n] e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} a[n] \cos(n\omega) - j \sum_{n=0}^{N-1} a[n] \sin(n\omega). \end{aligned} \quad (1)$$

Let

$$T_1 = \sum_{n=0}^{N-1} a[n] \cos(n\omega) \quad (2)$$

$$T_2 = \sum_{n=0}^{N-1} a[n] \sin(n\omega), \quad (3)$$

then the phase response is $\phi(\omega) = -\tan^{-1}(T_2/T_1)$ and the group delay is

$$\tau(\omega) = \frac{d}{d\omega}\phi(\omega) \quad (4)$$

$$= \frac{T_1 T_3 + T_2 T_4}{T_1^2} \cos^2 [\tan^{-1}(T_2/T_1)]. \quad (5)$$

In equation (5) we have

$$T_3 = \sum_{n=0}^{N-1} na[n] \cos(n\omega) \quad (6)$$

$$T_4 = \sum_{n=0}^{N-1} na[n] \sin(n\omega), \quad (7)$$

Define $\mathbf{a} = [a[0] \ a[1] \ \dots \ a[N-1]]^T$, $\mathbf{c}(\omega) = [1 \ \cos(\omega) \ \dots \ \cos((N-1)\omega)]^T$ and $\mathbf{s}(\omega) = [0 \ \sin(\omega) \ \dots \ \sin((N-1)\omega)]^T$. Thus equation (1) can be re-written as

$$H(\omega) = \mathbf{a}^T \mathbf{c}(\omega) - j\mathbf{a}^T \mathbf{s}(\omega) = H_r(\omega) - jH_i(\omega). \quad (8)$$

The goal of course is to approximate a desired filter $D(\omega)$ as closely as possible by adjusting the elements of \mathbf{a} . $D(\omega)$ could be an ideal low-pass filter, band-pass filter etc. And ‘‘closely’’ also requires definition. Let us start with D .

$$D(\omega) = \begin{cases} M(\omega) \cos p(\omega) - jM(\omega) \sin p(\omega) = D_r(\omega) - jD_i(\omega) & \omega \in \text{passbands} \\ 0 & \omega \in \text{stopbands} \end{cases} \quad (9)$$

As far as error goes, we consider the real and imaginary error separately and combine them as a sum. The real and imaginary errors in the passbands are:

$$\begin{aligned} E_r &= \left[\frac{D_r(\omega)}{D_r(\omega_0)} H_r(\omega_0) - H_r(\omega) \right]^2 = \mathbf{a}^T \left[\frac{D_r(\omega)}{D_r(\omega_0)} \mathbf{c}(\omega_0) - \mathbf{c}(\omega) \right] \left[\frac{D_r(\omega)}{D_r(\omega_0)} \mathbf{c}^T(\omega_0) - \mathbf{c}^T(\omega) \right] \mathbf{a} \\ E_i &= \left[\frac{D_i(\omega)}{D_i(\omega_0)} H_i(\omega_0) - H_i(\omega) \right]^2 = \mathbf{a}^T \left[\frac{D_i(\omega)}{D_i(\omega_0)} \mathbf{s}(\omega_0) - \mathbf{s}(\omega) \right] \left[\frac{D_i(\omega)}{D_i(\omega_0)} \mathbf{s}^T(\omega_0) - \mathbf{s}^T(\omega) \right] \mathbf{a}. \end{aligned} \quad (10)$$

Note terms like $D(\omega_0)$ and $H(\omega_0)$ in equation (10). ω_0 is a reference frequency at which $D(\omega_0) \neq 0$ and $E = 0$. Usually, ω_0 is placed in the middle of the passband. Also note when we have an error function of this form, we can write equation (10) in a quadratic form. After the relevant substitutions we get

$$\begin{aligned} E_r &= \mathbf{a}^T \left[\frac{M(\omega) \cos p(\omega)}{M(\omega_0) \cos p(\omega_0)} \mathbf{c}(\omega_0) - \mathbf{c}(\omega) \right] \left[\frac{M(\omega) \cos p(\omega)}{M(\omega_0) \cos p(\omega_0)} \mathbf{c}(\omega_0) - \mathbf{c}(\omega) \right]^T \mathbf{a} \\ E_i &= \mathbf{a}^T \left[\frac{M(\omega) \sin p(\omega)}{M(\omega_0) \sin p(\omega_0)} \mathbf{s}(\omega_0) - \mathbf{s}(\omega) \right] \left[\frac{M(\omega) \sin p(\omega)}{M(\omega_0) \sin p(\omega_0)} \mathbf{s}(\omega_0) - \mathbf{s}(\omega) \right]^T \mathbf{a}. \end{aligned} \quad (11)$$

The total error in the passband is

$$E_p = \int_{\omega_{p1}}^{\omega_{p2}} (E_r + E_i) d\omega \quad (12)$$

$$= \mathbf{a}^T \mathbf{Q}_p \mathbf{a} \quad (13)$$

\mathbf{Q}_p is a real-symmetric, positive-definite matrix. Through similar manipulations the stopband error is

$$\begin{aligned} E_s &= \mathbf{a}^T \left\{ \int_{\omega_{s1}}^{\omega_{s2}} [\mathbf{c}(\omega) \mathbf{c}^T(\omega) + \mathbf{s}(\omega) \mathbf{s}^T(\omega)] d\omega \right\} \mathbf{a} \\ &= \mathbf{a}^T \mathbf{Q}_s \mathbf{a} \end{aligned} \quad (14)$$

\mathbf{Q}_s is also a real-symmetric, positive-definite matrix while $\omega_{p_1}, \omega_{p_2}$ and $\omega_{s_1}, \omega_{s_2}$ are the edge frequencies of the passband and stopband respectively.

The total error can be written as a weighted sum of E_p and E_s , i.e.,

$$E = \alpha E_p + \beta E_s = \mathbf{a}^T [\alpha \mathbf{Q}_p + \beta \mathbf{Q}_s] \mathbf{a} \quad (15)$$

Let $\mathbf{Q} = \alpha \mathbf{Q}_p + \beta \mathbf{Q}_s$, then the total error is:

$$E = \mathbf{a}^T \mathbf{Q} \mathbf{a}, \quad (16)$$

where \mathbf{Q} is a real-symmetric and positive-definite matrix. If you expand out the value for E , you will see that it is quadratic in terms of the elements of \mathbf{a} . Note that E represents the mean square error between the ideal (desired) filter and the actual filter, so *minimizing* it makes sense. Also note, that all the elements of \mathbf{Q} are completely defined so when we speak of minimization, it is the \mathbf{a} 's that will be minimized. The fact that we have a quadratic form permits us to use Rayleigh's principle¹ for the minimization. This is stated as:

Theorem 1 *Let \mathbf{Q} be a symmetric $(N-1) \times (N-1)$ matrix with ordered eigenvalues $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{N-1}$. Given any $(N-1) \times 1$ vector \mathbf{a} , and the quadratic system $\mathbf{a}^T \mathbf{Q} \mathbf{a}$, the minimum \mathbf{a}_{min} of the quadratic system is attained at the eigenvalue of λ_{N-1} and is given by the eigenvector of \mathbf{Q} corresponding to λ_{N-1} . A restriction is that $\mathbf{a}^T \mathbf{a} = 1$.*

2 An example

We are going to design by hand a 3-tap FIR filter with passband is $0 \leq \omega \leq \pi/4$ ($\omega_{p_1} = 0, \omega_{p_2} = \pi/4$) and stopband $\pi/2 \leq \omega \leq \pi$ ($\omega_{s_1} = \pi/2, \omega_{s_2} = \pi$). From our choice of $\omega_{p_1}, \omega_{p_2}$ we know that ω_0 should lie between them and we choose $\omega_0 = \pi/8$. We desire a boxcar filter with cutoff frequency at $\omega_c = \pi/4$:

$$D(\omega) = \begin{cases} \cos \tau\omega - j \sin \tau\omega = e^{-j\tau\omega} & 0 \leq \omega \leq \pi/4 \\ 0 & \pi/2 \leq \omega \leq \pi \end{cases} \quad (17)$$

The student is asked to prove that the general term in the \mathbf{Q} matrix ($0 \leq n, m \leq N-1$) is:

$$\begin{aligned} q(n, m) &= \alpha \int_{\omega_{p_1}}^{\omega_{p_2}} [d_1 d_2 + d_3 d_4] d\omega + \beta \int_{\omega_{s_1}}^{\omega_{s_2}} (\cos n\omega \cos m\omega + \sin n\omega \sin m\omega) d\omega \\ d_1 &= \frac{\cos \tau\omega}{\cos \tau\omega_0} \cos n\omega_0 - \cos n\omega \\ d_2 &= \frac{\cos \tau\omega}{\cos \tau\omega_0} \cos m\omega_0 - \cos m\omega \\ d_3 &= \frac{\sin \tau\omega}{\sin \tau\omega_0} \sin n\omega_0 - \sin n\omega \\ d_4 &= \frac{\sin \tau\omega}{\sin \tau\omega_0} \sin m\omega_0 - \sin m\omega \end{aligned} \quad (18)$$

With $q(n, m)$ as above and assuming $\alpha = \beta = 1$ we have

$$\mathbf{Q} = \begin{pmatrix} 1.5874 & -1.0000 & -0.0190 \\ -1.0000 & 1.5708 & -1.0000 \\ -0.0190 & -1.0000 & 1.6272 \end{pmatrix} \quad (19)$$

From \mathbf{Q} and Theorem 1 we get the FIR filter $\mathbf{a} = [0.5072 \ 0.7086 \ 0.4095]^T$

¹A good reference for this is *Matrix Analysis for Statistics*, James. R. Schott, Theorem 3.15, page 105, Wiley.

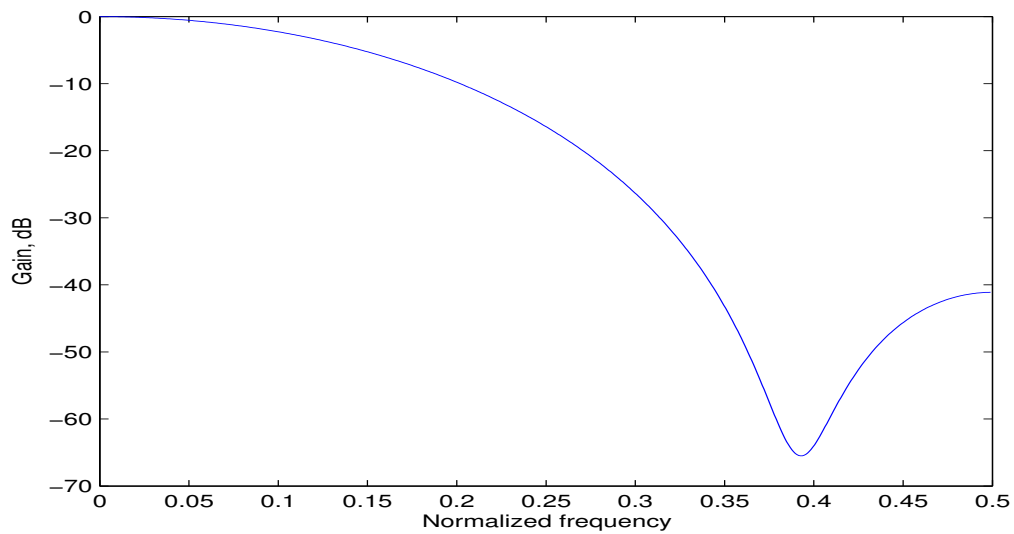


Figure 1: Gain of the FIR filter $\mathbf{a} = [0.5072 \ 0.7086 \ 0.4095]^T$. Note that the gain is not constant in the passband frequencies $[\omega_{p1} \ \omega_{p2}] = [0 \ \pi/4]$

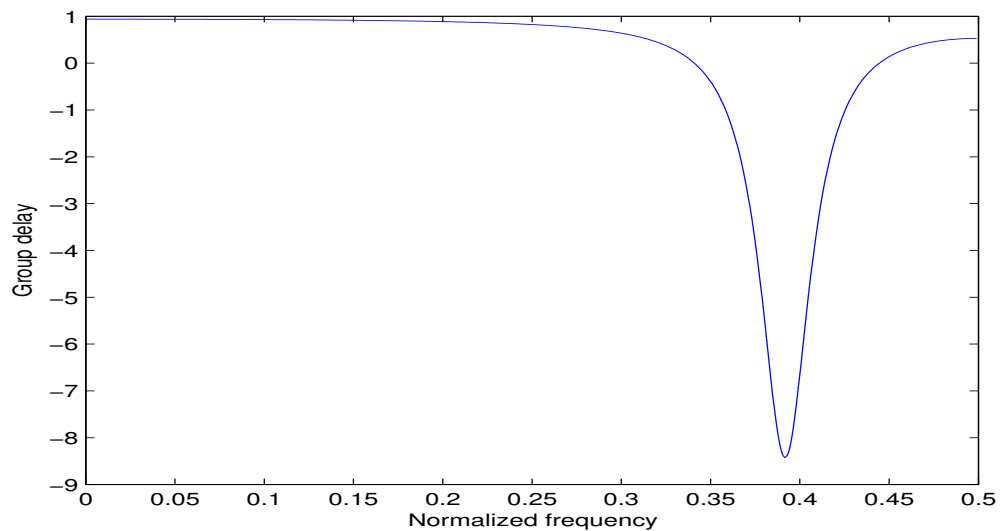


Figure 2: Group of the FIR filter $\mathbf{a} = [0.5072 \ 0.7086 \ 0.4095]^T$. The group delay is more or less constant in the passband frequencies $[\omega_{p1} \ \omega_{p2}] = [0 \ \pi/4]$

3 The full-blown problem

The previous problem deliberately chosen because the calculations were feasible by hand. Let us proceed to a larger problem which can only be solved on a computer.

Design a low-pass filter with $N = 31$, $f_p = 0.06$ and $f_s = 0.12$. We wish to have a group delay of $\tau = 12$ in the passband. Use $\alpha = 1, \beta = 5$ in the design and assume that the filter has zero error $\omega_0 = f_p/2$.

4 What to do?

This project has two parts. The first part deals with performing derivations in order to understand the basic theory while the latter one deals with programming in MATLAB.

Derivations - 50%.

1. Prove the first of the pair of equations given by equation (14).
2. Using equations (13, 14) prove equation (18) and then prove that \mathbf{Q} is symmetric.
3. Show the steps in the calculations that enable you to get numerical values for equation (18) as shown in equation (19)
4. **Due date:** 8 May 2006 in class or by email (if students use a word processor). Your submissions should be neatly written (or typed). Make all assumptions clear to the reader and show all steps in derivations. If you are using tables of integrals, copy the integral rather than make a reference to it.

MATLAB - 50%

- Write a MATLAB program (not a function) that solves the full blown problem. Your program will produce the following outputs
 1. The matrix \mathbf{Q} using the `disp` command in MATLAB. Also be sure to use `format long` so that you have 16 digits of output.
 2. The vector \mathbf{a} or \mathbf{a}^T .
 3. The gain and group delay corresponding to \mathbf{a} .
- **Due date:** 15 May 2006 by email only.