

An Example of a Theorem that has Contradictory Relativizations and a Diagonalization Proof

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Introduction

The central questions in complexity theory (e.g. the $P = ?NP$ question) can only be solved with proof techniques that do not relativize [BGS75]. There had been some debate about whether such techniques are within reach of the “current state of mathematics” [Hop84]. Recent advances in the area of interactive protocols have produced techniques that do not relativize [Sha90, LFKN90, HCRR90]. However, these advances do not resolve the debate on whether *diagonalization* can be used to solve problems that have contradictory relativizations [BGS75, Koz80]. In the following, we give an example of a theorem that has contradictory relativizations *and* a diagonalization proof.

The Example

Theorem. There exists a computable space bound $S(n)$ and an oracle A such that, $n^2 \leq S(n) < n^3$,

$$\text{DSPACE}[S(n)] = \text{DSPACE}[S(n) \log n],$$

but

$$\text{DSPACE}^A[S(n)] \neq \text{DSPACE}^A[S(n) \log n].$$

Proof: We construct $S(n)$ so that no Turing machine uses space between $S(n)$ and $S(n) \log n$. At step n of the construction, we compute $S(n)$ by diagonalizing over a slowly growing list of Turing machines which use at most n^3 space. We make this list of machines grow so slowly that it contains only the first $r(n) - 1$ machines, where

$$r(n) = \frac{\log n}{\log \log n}.$$

Consider the following function:

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$$f(i, n) = n^2 \log^i n.$$

Note that $f(i+1, n) = f(i, n) \log n$ and that $f(r(n), n) \leq n^3$. For each i , $0 \leq i < r(n)$, $f(i, n)$ is possible value for $S(n)$. The only requirement we need to satisfy is that no machine on the list uses space between $f(i, n)$ and $f(i, n) \log n$. Since there are only $r(n) - 1$ machines, by the Pigeonhole Principle, there must be some i_0 , $0 \leq i_0 < r(n)$ such that no machine's space bound falls between $f(i_0, n)$ and $f(i_0 + 1, n) = f(i_0, n) \log n$. Let $S(n) = f(i_0, n)$. To find this particular i_0 , simply compute and record the maximum amount of space that each machine uses for any input of length n . (This is computable since the machines are space bounded.) Then, find the first i_0 that satisfies the requirement.

Thus, every machine which uses less than n^3 space either uses less than $S(n)$ space cofinitely often or uses greater than $S(n) \log n$ space infinitely often. So, $\text{DSPACE}[S(n)] = \text{DSPACE}[S(n) \log n]$. This construction does not violate the Space Hierarchy Theorem [HU79, SHL65] because while $S(n)$ is easily computable in PSPACE, it is not fully space constructible.¹ In fact, this theorem is an application of the Gap Theorem [Bor72, Tra67].

Now, consider the oracle

$$A = \{k\#n \mid k \leq S(n)\}.$$

$S(n)$ becomes space constructible with A as an oracle. So, by the Space Hierarchy Theorem,

$$\text{DSPACE}^A[S(n)] \neq \text{DSPACE}^A[S(n) \log n]. \quad \square$$

Two observations are worth noting. First, the construction above works just as well for nondeterministic space, deterministic time and nondeterministic time. For the nondeterministic cases, the oracle A will also provide a language which diagonalizes over $\text{NSPACE}^A[S(n)]$ and $\text{NTIME}^A[S(n)]$. Second, $S(n)$ constructed above may not be monotonic. However, the same construction restricting $S(n)$ to be between 2^{n^2} and $2^{(n+1)^2}$ will give a monotonic space bound with the same properties.

References

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¹A space bound $S(n)$ is fully space constructible if there exists a Turing machine that uses exactly $S(n)$ tape cells on input 1^n .

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