THE COMBINATORICS OF POKER

Poker is a card game in which players are dealt a number of cards, usually 5 or 7, and wager that the hand they have will beat the hand that any other player has. The determination that one hand is better than another is made by comparing the probabilities that each hand will occur and judging that hands with smaller probabilities are more valuable than hands with bigger ones.

To find the probability of each hand, we have to count the number of distinct ways that hand can happen and divide it by the total number of hands. We will concentrate on 5-card draw poker for this study.

The standard deck of poker cards is just the standard deck of 52 cards, which are subdivided as 4 suits – Clubs (♣), Diamonds (♦), Hearts (♥), and Spades (♠), with 13 values in each suit (2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, and Ace). So, in 5-card draw poker, there are: $C(52,5) = 52!/(5!\,47!) = 2{,}598{,}960$ distinct hands.

Now the first special hand is a **PAIR**, made up of 2 cards of the same value and the other 3 mixed values; for example the hand 8♣8♦A♣Q♦4♠ is a pair of 8's with "Ace high," to indicate the next most valuable card in comparing with another player who might also have a pair of 8's. How many ways are there to get a hand with one pair? As we shall see this is not hard to count, but there are also alternate ways to **ATTEMPT** to model this event which will lead to an incorrect accounting!

CORRECT METHOD: The problem of dealing a pair is broken down as:

(pick a value)(choose 2 suits to have that value)(pick the remaining 3 values)(pick the suits for the remaining values)

so the counts are:

$$C(13,1)C(4,2)C(12,3)C(4,1)C(4,1)C(4,1) = (13\cdot4\cdot12\cdot11\cdot10\cdot4\cdot4\cdot4)/(2\cdot3\cdot2) = 1{,}098{,}240 \text{ ways.}$$

INCORRECT METHOD: After we count the pair, if we think to fill out the last three cards as:

(pick any remaining card)(pick any remaining card that doesn't match the last 2 values)(pick any remaining card that doesn't match the last 3 values)

the counts for this part are: $48\cdot44\cdot42$, but this now treats having A♣Q♦4♠, Q♦4♠A♣, 4♠A♣Q♦, Q♦A♣4♠, 4♠Q♦A♣, etc., as different events. Clearly this multiple counting will yield an incorrect solution. We note here, that the error in judgement was that we treated the finding of the last three cards as an *ordered* event rather than an *unordered* one.

Try to find the counts for the other special poker hands as described below:

2 PAIR (i.e. 8♣8♦A♣A♦4♠)
3 OF A KIND (i.e. 8♣8♦8♠Q♦4♠, but no full houses allowed!)
STRAIGHT (i.e. 8♣9♦10♣J♦Q♠, but no flushes allowed!)
FLUSH (i.e. A♣Q♣8♣4♣3♣, but no straights allowed!)
FULL HOUSE (i.e. 8♣8♦A♣A♦A♠)
4 OF A KIND (i.e. 8♣8♦8♥8♠4♠)
STRAIGHT FLUSH (i.e. 8♣9♣10♣J♣Q♣, but no royal flushes allowed!)
ROYAL FLUSH (i.e. 10♣J♣Q♣K♣A♣)

ANSWERS:

2 Pair: $C(13,2)C(4,2)C(4,2)C(44,1) = 123{,}552$ hands.
3 Of A Kind: $C(13,1)C(4,3)C(12,2)C(4,1)C(4,1) = 54{,}912$ hands.
Straight: $C(9,1)(4^5 - 4) = 9{,}180$ hands.
Flush: $C(4,1)[C(13,5) - C(9,1)] = 5{,}112$ hands.
Full House: $C(13,1)C(4,3)C(12,1)C(4,2) = 3{,}744$ hands.
4 Of A Kind: $C(13,1)C(4,4)C(12,1)C(4,1) = 624$ hands.
Straight Flush: $C(8,1)C(4,1) = 32$ hands.
Royal Flush: $C(4,1) = 4$ hands.