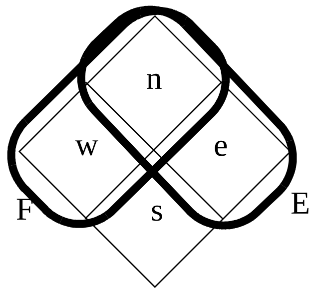
<u>Probabilistic Independence</u>

Consider the following drawing:



In this drawing, we see that $|E| = n + e$, $|F| = w + n$, $|E \cap F| = n$, and $|S| = n + s + e + w$.

Now, from Conditional Probability, we have that: $P(E \mid F) = P(E \cap F) / P(F)$, and
when E is independent of F, we have that: $P(E \mid F) = P(E)$, so combining these equations yields:

$$P(E) = P(E \mid F) = P(E \cap F) / P(F),$$

$$\text{hence } P(E)P(F) = P(E \cap F).$$

This is our test for Independence: $P(E \cap F) = P(E)P(F)$.

<u>What This Really Means:</u>

If we replace these terms using their equivalent cardinality forms:

$$P(E \cap F) = |E \cap F| / |S|, \; P(E) = |E| / |S|, \text{ and } P(F) = |F| / |S|$$

we get:

$$P(E \cap F) = |E \cap F| / |S| = (|E| / |S|)(|F| / |S|) = P(E)P(F)$$

so multiplying by $|S|^2$ yields:

$$|E \cap F| \times |S| = |E| \times |F|.$$

In terms of the n, s, e, and w regions, this becomes:

$$n(n + s + e + w) = (n + e)(n + w)$$

so

$$nn + ns + ne + nw = nn + ne + nw + ew$$

hence

$$ns = ew, \text{ which is equivalent to } n / w = e / s.$$

So, what is Independence of E with respect to F? The ratio (n / w) compares the part of E within F to the rest of F, and the ratio (e / s) compares the part of E outside F to the rest of the things outside F.

<u>Conclusion:</u> E is independent of F when the ratio of E in F to the rest of F equals the ratio of E outside of F to the rest of the things outside of F. That is, E is equally likely within F or outside F.