Set Theory

If A and B are sets, A is called a **subset** of B, written **$A \subseteq B$**, if and only if ($\leftrightarrow$), every element of A is also an element of B.

Symbolically:

$A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$.

The phrases A *is contained in B and B contains A* are alternative ways of saying that A is a subset of B.

Let A and B be sets. A is a **proper subset** of B, if and only if ($\leftrightarrow$), every element of A is in B but there is at least one element of B that is not in A.

Given sets A and B, A equals B, written A = B, $\leftrightarrow A \subseteq B \wedge B \subseteq A$

Let A and B be subsets of the universal set U:

The **union** of A and B is the set of all elements x in U such that x is in A or X is in B.

$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$ note this is an "or", not an XOR $\oplus$

The **intersection** of A and B is the set of all elements x in U such that x is in A and X is in B.

$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$

The **difference** of B minus A (or **relative complement** of A in B), is the set of all elements x in U such that x is in B and x is not in A.

$B - A = \{x \in U \mid x \in B \wedge x \notin A\}$

The complement of A, is the set of all elements x in U such that x is not in A.

$A^c = \{x \in U \mid x \notin A\}$

Cartesian Products

The **ordered *n*-tuple** $(x_1, x_2, x_3, \ldots x_n)$ consists of $x_1, x_2, \ldots$ in a specific order up to xn. An ordered 2-tuple is called an **ordered pair**, an ordered 3 tuple is called an **ordered triple**. Ordered n-tuples are **equal** if and only if

$x_1 = y_1 = z_1 =$ etc; $x_2 = y_2 = z_2 =$ etc. As shown in the following ordered pair:

$(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.

The **Cartesian product** of A and B, denoted A X B (read "A cross B"), is the set of all ordered pairs (a,b), where *a* is in A and *b* is in B.

$\{(x,y) \in A \times B \mid x \in A \wedge y \in B\}$

Formal Languages

The **alphabet**, the finite set of symbols, denoted by the letter Σ. A sting of characters of an alphabet Σ (or a **string over Σ**) is either (1) and ordered *n*-tuple of elements written without parenthese or commas, or (2) the null string ε, which has no characters. The length of a string is the number of characters that make up the alphabet.

Notation:

Let be an alphabet. For each non negative integer n, let

Σ^n = the set of all strings over that have length n, and

Σ^* = the set of all strings of finite lengh over Σ.

Subset Relations

Inclusion of Intersection: $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Inclusion in Union: $A \subseteq A \cup B$ and $B \subseteq A \cup B$

Transitive Property of Subsets: if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Set Identities

Communative Law: $A \cap B = B \cap A$ and $A \cup B = B \cup A$

Associative Law: $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Intersection with U: $A \cap U = A$

Union with U: $A \cup U = U$

Double Complement: $(A^c)^c = A$

Idepotent Laws: $A \cap A = A$ and $A \cup A = A$

DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Asorbtion Laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$

Alternate Representation of diff: $A - B = A \cap B^c$

The unique set with no elements is called the empty set: $\varnothing$

Two sets are disjoint $\leftrightarrow$ they have no elements in common: $A \cap B = \varnothing$

Mutually Disjoint; Pairwise Disjoint or **Non-overlapping** sets: $\forall$ i, j $A_i \cap A_j = \varnothing$ whenever $i \neq j$. A partition exist if a union of all Sets A equals A and all Sets A are mutually Disjoint.

Power Set of A: Given set A, the **power set**, $\wp(A)$, is the set of all subsets of A.

For all sets A and B, if $A \subseteq B$ then $\wp(A) \subseteq \wp(B)$.

For all integers n 0, if a set X has n elements then $\wp(X)$ has 2^n elements.

Boolean Algebra: substitute $\vee$ for $\cup$ and $\wedge$ for $\cap$ in the above equations.

Complement Laws: $a + a' = 1$ and $a \cdot a' = 0$.

Proving Two sets, X and Y are Equal

Let sets X and Y be given. To prove $X = Y$:

1. prove that $X \subseteq Y$.
2. prove that $Y \subseteq X$

Empty Set, Partitions, Power Sets and Boolean Algebra

A set with no elements is a subset of Every set.

If $\varnothing$ is a set with no elements and A is any set then $\varnothing \subseteq A$.

Uniqueness of the Empty Set.

There is only one set with no elements, the Empty Set.

Properties that involve $\varnothing$.

Let all sets referred to below be subsets of a universal set U.

1. Union : $A \cup \varnothing = A$
2. Intersection and Union with the Complement:

   $A \cap A^c = \varnothing$ and $A \cup A^c = U$
3. Intersection with $\varnothing$, Acts as a Universal Bound for $\cap$, $A \cap \varnothing = \varnothing$
4. Complement of U and $\varnothing$: $U^c = \varnothing$ and $\varnothing^c$ U

Problem Solving Strategies
Optimistic: Directly prove the statement, "What do I need to show and how to show it?"
Pessimistic: Look for a set of conditions that must be fullfilled to show a counterexample.

Formal Logic
A statement (or **Proposition**) is a sentance that is true or false but not both.

Conjunction of p and q: $p \wedge q$ (1 if both p and q are true)
Disjunction of p and q: $p \vee q$ (1 if either p or q or both are true)
Negation of p: $\sim p$ (1 if p is 0, 0 if p is 1)

Two *statement forms* are **logically equivelent**, if and only if they have identical truth values for each possible substitution of statements for their statements variable: $P \equiv Q$.
Two *statements* are **logically equivelent** $\leftrightarrow$ with the same statement variables, the forms are logically equivelent.

DeMorgan's Laws
The negation of an ***and*** statement is logically equivelent to the ***or*** statement in which each component is negated.
The negation of an ***or*** statement is logically equivelent to the ***and*** statement in which each component is negated.

Tautology is always true. $p \vee \sim p \equiv t$
Contradiction is always false. $p \wedge \sim p \equiv c$

Logical Equivalences Given p, q and r; tautology t aka 1; and contradiction c aka 0:
Commutative laws: $p \wedge q \equiv q \wedge p$ $\quad p \vee q \equiv q \vee p$
Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\quad (p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws: $p \wedge t \equiv p$ $\quad p \vee c \equiv p$
Negation laws $p \vee \sim p \equiv t$ $\quad p \wedge \sim p \equiv c$
Double negative: $\sim(\sim p) \equiv p$
Indempotent laws: $p \wedge p \equiv p$ $\quad p \vee p \equiv p$
De Morgan's laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$ $\quad \sim(p \vee q) \equiv \sim p \wedge \sim q$
Universal Bound: $p \vee t \equiv t$ $\quad p \wedge c \equiv c$
Asorbtion : $p \vee (p \wedge q) \equiv p$ $\quad p \wedge (p \vee q) \equiv p$
Negation of t and c: $\sim t \equiv c$ $\quad \sim c \equiv t$

Conditional Statements
The **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \rightarrow q$
p is the *hypotheses*; q is the *conclusion*

logical equivelence:
$p \rightarrow q \equiv \sim p \vee q$
Reason that they are equivelent is that if p is false, you cannot determine if q is false or true. The relationship addresses only one of four situations: if p is true, then q is true. The statement can only be false if q can be false with p being true. This is the negation of the statement (see below).

Negation of a conditional statement:
$\sim(p \rightarrow q) \equiv p \wedge \sim q$

The **contrapositive** of a conditional statement $p \to q$ is $\sim q \to \sim p$ The positive and contrapositive statement are logically equivelant.

Converse and Inverse of $p \to q$: **converse** is $q \to p$ **inverse** is $\sim p \to \sim q$

Converse and Inverse statements are logically equivelant to each other, but not to the original conditional statement.

Only if and the **biconditional**: p **only if** q means "if not q then not p." The biconditional is "p **if, and only if,** q" denoted by $p \leftrightarrow q$. This is also abreviated **iff**. It is true if both p and q are true **OR** both are false.

positive statements: $p \to q$ if r(x) then s(x)

Necessary and sufficient conditions: p is a **sufficient condition** for q means if p then q and $\sim q$ then $\sim p$.
and

$\forall$ x, if r(x) then s(x) **only if** has the same logic equations but emphasizes $\sim s(x)$ then $\sim r(x)$ and $\sim q$ then $\sim p$

p is a **necessary condition** for q means if not p then not q. $\forall$ x, if $\sim r(x)$ then $\sim s(x)$ **or** $\forall$ x, if s(x) then r(x)

An **argument** is a sequence of statements. All statements but the final one are called **premises, assumptions** or **hypotheses**. The final statement is called the **conclusion**. The symbol for therefore, $\therefore$, is placed before the conclusion.

An *argument form* is **valid** if no matter what particular statements are substituted for the statement variables in its premises, if they are all true, then the conclusion is also true.
An *argument* is **valid** means its form is valid.

To test for validity of an argument:

1. Identify the premises and conclusion
2. Construct a truth table for all premises and the conclusion
3. Find the **critical rows** in which all the premises are true
4. In each critical row, determine whether the conclusion is also true. If in any of the critical rows the conclusion is false, the *argument form* is false. Do sanity check with data used to establish the premisis to check for False.

Modus ponens: argument by affirming

$p \to q$
p
$\therefore$ q

Modus tollens: argument by denying

$p \to q$
$\sim q$
$\therefore \sim p$

Disjunctive addition (if know answer to one half of an "or" statement is true, do not have to ask about other half): given p (or q) is true, $\therefore$ $p \vee q$ is true.

Conjunctive simlification:

$p \wedge q$
$\therefore$ p (or q) for first statement both p & q had to be true, which implies 2nd statement.

Conjunctive addition:

p
q
$\therefore$ $p \wedge q$

Disjunctive Syllogism:
$p \vee q$
~q (or ~p)
$\therefore$ p (or q)

Hypothetical Syllogism
$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

Proof by division into cases: Given an ambigous beginning, the outcome is the same:
$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore$ r

Rule of contradiction
$\sim p \rightarrow c$
$\therefore$ p

Two digital logic circuits are *equivalent* if, and only if, their input/output tables are identical.

Fallacies
The **converse error**: Tries to do a reverse Modus Ponen
$p \rightarrow q$
q
$\therefore$ p   This is not an **iff** situation, because q exists doesn't mean it needed p to exist.

The **inverse error**: Tries to do a reverse Modus Tollen
$p \rightarrow q$
~p
$\therefore$ ~q   This is not an **iff** situation, because p is false doesn't mean q is false.

- Valid Argument with a False Conclusion: good form, but premise is false. The example given is John Lennon's status of rock star determined hair color.
- Invalid argument with a true conclusion, truth is despite faulty reasoning.

Remember a conditional statement is not logically equivelant to its converse or inverse.

Contradiction rule: If a supposition that p is false leads logically to a contradiction, then p is true.
$\sim p \rightarrow c$ aka 0
$\therefore$ p

Statement calculus no allowance for "all" or "some" etc.
Predicate calculus: analyzes quantities
Predicate: partial sentence with variables, becomes a statement when variables are given values.
Domain of predicate variable: set of all values that variable can be.

If P(x) is a predicate with domain D, truth set is $\{x \in D \mid P(x)\}$

universal statement: $\forall$ x $\in$ D, P(x) is true as long as there is no **counterexample**.

An **existential statement** $\exists$x $\in$ D, P(x) true as long as there is one or more true examples.

universal conditional statement: $\forall$ x, if P(x) then Q(x)

negation of universal statement:
~($\forall$x $\in$ D, P(x)) $\equiv$ $\exists$x $\in$ D such that ~P(x)
The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not")

negation of an existential statement:
~($\exists$ x $\in$ D such that P(x)) $\equiv$ $\forall$ x $\in$ D, ~P(x)
The negation of an existential statement ("some are") is logically equivelent to a universal statement ("all are not")

negation of universal conditional statements:
~($\forall$ x, if P(x) then Q(x)) $\equiv$ $\exists$ x such that P(x) and ~Q(x)

Negations of statements:

No politician is honest.
formal version: $\forall$ politicians *x*, *x* is not honest.
formal negation: $\exists$ a politician *x*, such that *x* is honest.
informal negation: Some politicians are honest.

Illustration of a false premise leading to a true every time statement:
An empty fruit bowl, and the statement "All fruit in the bowl are oranges."
The negation is "There is a piece of fruit in the bowl that is not an orange."
The only way for that statement to be true would be if there were a piece of fruit in the bowl.
Since it is impossible to negate the statement, the statement is **true by default**, or **vacuously true**.
$\forall$ *x* in D, if P(*x*) then Q(*x*) is **true by default**, or **vacuously true** iff P(*x*) is false for every *x* in D.

Universal instantiation

- If some property is true of true of everything in a domain, then it is true of any particular thing in the domain.

VARIANTS OF UNIVERSAL CONDITIONAL STATEMENTS
positive statement: $\forall$ *x* $\in$ D, if P(*x*) then Q(*x*)

1. Its **contrapositive** is the statment
   $\forall$ *x* $\in$ D, if ~Q(*x*) then ~P(*x*)
2. Its **converse** is the statment
   $\forall$ *x* $\in$ D, if Q(*x*) then P(*x*)
3. Its **inverse** is the statment
   $\forall$ *x* $\in$ D, if ~P(*x*) then ~Q(*x*)

Universal Modus Ponens

Formal Version	Informal Version
$\forall x$, if P(x) then Q(x)	if x makes P(x) true, then x makes Q(x) true.
P(a) for a particular a	a makes P(x) true
$\therefore$ Q(a)	$\therefore$ a makes Q(x) true

Universal Modus Tollens

Formal Version	Informal Version
$\forall x$, if P(x) then Q(x)	if x makes P(x) true, then x makes Q(x) true.
~Q(a) for a particular a	a does not make Q(x) true
$\therefore$ ~P(a)	$\therefore$ a does not make P(x) true

Converse Error (Quantified Form)

The following argument form is *invalid*:

Formal Version	Informal Version
$\forall x$, if P(x) then Q(x)	if x makes P(x) true, then x makes Q(x) true.
Q(a) for a particular a	a makes Q(x) true
$\therefore$ P(a) invalid conclusion	$\therefore$ a makes P(x) true invalid conclusion

Inverse Error (Quantified Form)

The following argument form is *invalid*:

Formal Version	Informal Version
$\forall x$, if P(x) then Q(x)	if x makes P(x) true, then x makes Q(x) true.
~P(a) for a particular a	a does not make P(x) true
$\therefore$ ~Q(a) invalid conclusion	$\therefore$ a does not make Q(x) true invalid conclusion

VARIANTS OF UNIVERSAL CONDITIONAL STATEMENTS

positive statement: $\forall x \in$ D, if P(x) then Q(x)

1. Its **contrapositive** is the statment
   $\forall x \in$ D, if ~Q(x) then ~P(x)
2. Its **converse** is the statment
   $\forall x \in$ D, if Q(x) then P(x)
3. Its **inverse** is the statment
   $\forall x \in$ D, if ~P(x) then ~Q(x)

Method of generalizing from the generic particular

- To show that every element of a domain satisfies a certain property, suppose x is a particular but arbitrarily chosen element of the domain and show that x satisfies the property.

Method of Direct Proof of the relationship *P(x) → Q(x)*

1. Express the statement to be proved in the form "$\forall x \in$ D, if P(x) then Q(x)" (only if Q(x) can be shown to be true, can the statement be proven to be true)
2. Start the proof by supposing that x is a particular but arbitrarily chosen element of D for which the hypothesis P(x) is true, abbreviated "Suppose $x \in$ D and P(x)."
3. Show that the conclusion Q(x) is true by using definitions, previously established results, and the rules for logical inference.

Informally: *supposing P(x) is true, is Q(x) true?*

1. **Write theorum to be proved.**
2. **Clearly mark beginning of proof with the word *Proof.***
3. **Make proof self contained.**

Common Mistakes

1. Arguing from examples.
- a universal statement may be true in many instances without being true in all instances.

2. Using the same variable to mean two different things.
- Leads to the conclusion that two different things are the same.

3. Jumping to a conclusion.
- Allege the truth of something without an adequate reason.

4. Begging the question.
- A variation of jumping to a conclusion, assuming that what is to be proven is true (or false), and using that assumption as the proof.

5. Misuse of the word *if.*
- Using the word *if* when the word *because* or *since* is really meant. When the concept is not in doubt, use *because* or *since.*

Method of Proof by Contradiction

1. Suppose the statement to be proved is false.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Supposing a statement is false is the same as supposing the negation is true.

Method of Proof by Contraposition

1. Express the statement to be proved in the form "$\forall x \in D$, if $P(x)$ then $Q(x)$"
2. Rewrite the statement in the contrapositive form "$\forall x \in D$, if $Q(x)$ is false then $P(x)$ is false."
3. Prove the contrapositve by direct proof.:
   - Suppose x is a particular but arbitrarily chosen element of D such that $Q(x)$ is false.
   - Show that $P(x)$ is false.