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CMSC 203 - Homework Assignment 3 - Due April 20, 2011

1. Find the terms a_3, a_4, a_5, and a_6 for the recursively defined sequence given by: $a_0 = -1$, $a_1 = 0$, $a_2 = 1$, and $a_n = 2(a_{n-1})(a_{n-3}) + (a_{n-2})^2$ for $n \geq 3$.

$a_3 = 2(a_2)(a_0) + (a_1)^2 = 2(1)(-1) + 2(0)^2 = -2 + 0 = -2$

$a_4 = 2(a_3)(a_1) + (a_2)^2 = 2(-2)(0) + (1)^2 = 0 + 1 = 1$

$a_5 = 2(a_4)(a_2) + (a_3)^2 = 2(1)(1) + (-2)^2 = 2 + 4 = 6$

$a_6 = 2(a_5)(a_3) + (a_4)^2 = 2(6)(-2) + (1)^2 = -24 + 1 = -23$

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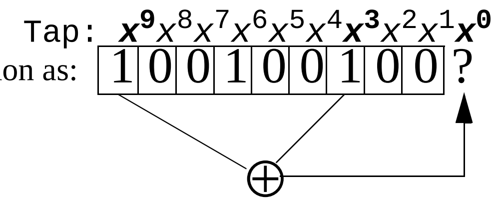
2. In the study of Coding Theory, we quickly encounter recursions on binary strings of data. In particular, the notion of a Linear Recursive Sequence (LRS) is very useful. Consider the following binary sequence definition:

1. Start with initial fill: 100100100.
2. For the bits in positions x^9 and x^3, calculate the bit in position x^0 as ($x^9 \oplus x^3$).
3. Shift one bit to the right and continue.

Typically, we draw this action as:



So, applying this polynomial to the initial "fill" 100100100 yields the next bit $1 \oplus 1 = 0$. Shifting the "taps" of the polynomial one position to the right and continuing, we get successive bits:
0, 0, 1, etc.

(a) Starting with the initial fill 1101000, apply the polynomial $x^7 \oplus x^2 \oplus x^0$ to generate the next 4 bits.

1101000: $1 \oplus 0 = 1$

1010001: $1 \oplus 0 = 1$

0100011: $0 \oplus 1 = 1$

1000111: $1 \oplus 1 = 0$

(b) Starting with the initial fill 1111, apply the polynomial $x^4 \oplus x^3 \oplus x^0$ until it "cycles".

P(1111) = 0, P(1110) = 0, P(1100) = 0, P(1000) = 1, P(0001) = 0, P(0010) = 0, P(0100) = 1,

P(1001) = 1, P(0011) = 0, P(0110) = 1, P(1101) = 0, P(1010) = 1, P(0101) = 1, P(1011) = 1,

P(0111) = 1, so the polynomial has generated the sequence 1111000100110101111.

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3. Using Mathematical Induction, prove for all Natural Numbers n, $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Proof: (Weak Induction)

Basis Step: We will show the statement holds for $n = 0$.

Now, $\sum_{i=0}^{0} i^3 = 0^3 = 0$ and $0^2(0+1)^2/4 = 0(1)/4 = 0/4 = 0$, hence $\sum_{i=0}^{0} i^3 = 0^2(0+1)^2/4$.

Inductive Step: Assume $\sum_{i=0}^{k} i^3 = \frac{k^2(k+1)^2}{4}$, for some $k > 0$ in the Natural Numbers. We want to show:

$$\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}.$$

Now, $[0^3 + 1^3 + 2^3 + ... + k^3 + (k+1)^3] = [0^3 + 1^3 + 2^3 + ... + k^3] + (k+1)^3$

$= [k^2(k+1)^2 / 4] + (k+1)^3$

$= [k^2(k+1)^2 / 4] + [4(k+1)(k+1)^2 / 4]$

$= [(k+1)^2 / 4][k^2 + 4(k+1)]$

$= [(k+1)^2 / 4](k^2 + 4k + 4)$

$= [(k+1)^2 / 4](k+2)^2$

$= (k+1)^2(k+2)^2 / 4.$

Therefore, $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, for all Natural Numbers n. QED

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4. Using the Strong Form of Mathematical Induction, prove for all Natural Numbers $n \geq 3$:

If $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$ and $a_0 = 0$, $a_1 = 3$ and $a_2 = 6$, then 3 divides a_n.

Proof: (Strong Induction)

Basis Step: Show 3 divides a_3.

Now, $a_3 = a_2 + 2a_1 + 3a_0 = 6 + 2(3) + 3(0) = 6 + 6 = 12 = 3(4)$, and 4 is an Integer, hence 3 divides a_3.

Inductive Step: Assume 3 divides each of the terms $a_4, a_5, a_6, ..., a_{k-1}, a_k$. We need to show 3 divides a_{k+1}.

Now, $a_{k+1} = a_k + 2a_{k-1} + 3a_{k-2}$. By the Inductive Hypothesis, we know that $3|a_k$, $3|a_{k-1}$ and $3|a_{k-2}$, hence there are Integers p, q, and r such that $a_k = 3p$, $a_{k-1} = 3q$, and $a_{k-2} = 3r$.

This implies $a_{k+1} = 3p + 2(3q) + 3(3r) = 3(p + 2q + 3r)$. Since p, q, and r are Integers, it follows that $(p + 2q + 3r)$ is an Integer, thus a_{k+1} is divisible by 3.

Therefore, 3 divides a_n for all Natural Numbers $n \geq 3$. QED

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5. Suppose I have 30 different math books, 20 different chemistry books and 10 different biology books.

(a) How many ways can I line all these books on shelf that is large enough to hold them all?

P(30 + 20 + 10 , 30 + 20 + 10) = P(60,60) = 60!

(b) How many ways can I line them up so that all the books of the same type are grouped together?

(Order subjects)(Order math books)(Order chemistry books)(Order biology books)

= 3! (30!) (20!) (10!)

(c) How many ways can I choose 3 of each to carry in a bookbag?

(Choose 3 math books)(Choose 3 chemistry books)(Choose 3 biology books)

= C(30,3)C(20,3)C(10,3)

= (30! / 27!3!)(20! / 17!3!)(10! / 7!3!).

(d) How many ways can I perform (c) if I do not want to have a certain pair of biology books together in any of the selections?

All - (Total with the certain pair together) = C(30,3)C(20,3)[C(10,3) - C(8,1)].

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6. The Mars Candy Company sells bags of M&Ms with 150 pieces candy colored from 10 different colors.

(a) How many different bags can they produce?

Categories = C = 10
Slots = S = 150
Total Restrictions = TR = 0

C((S + C – TR – 1), (S – TR)) = C((10 + 150 – 0 – 1), (150 – 0))

= C(159, 150)

= 159! / (150! 9!).

(b) How many different bags can they produce if each bag must contain at least 10 of each color?

Categories = C = 10
Slots = S = 150
Total Restrictions = TR = 10(10) = 100

C((S + C – TR – 1), (S – TR)) = C((10 + 150 – 100 – 1), (150 – 100))

= C(59, 50)

= 59! / (50! 9!).