CMSC 203 - Discrete Structures - FINAL EXAM - Fall 2010 **Name** SOLUTION KEY

1. (a) (3 pts.) Fill in the following Truth Tables.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(b) (3 pts.) Use the Laws of Logic to show: $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$.

$$(p \vee q) \to r \equiv \neg(p \vee q) \vee r \equiv (\neg p \wedge \neg q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$$

$$\equiv (p \to r) \wedge (q \to r).$$

(c) (3 pts.) Use the rules of inference to show the following is a valid argument:

$\neg p \vee q$
$\neg s \vee p$
$\neg s \to r$
$r \to u$
$\neg q$
$\therefore u$

$$\begin{array}{l} \neg p \vee q \\ \neg q \\ \hline \therefore \neg p \end{array} \qquad \begin{array}{l} \neg s \vee p \\ \neg p \\ \hline \therefore \neg s \end{array} \qquad \begin{array}{l} \neg s \to r \\ \neg s \\ \hline \therefore r \end{array} \qquad \begin{array}{l} r \to u \\ r \\ \hline \therefore u \end{array}$$

2. (a) (3 pts.) Find A × A × A for the set A = $\{x, y\}$.

$$A \times A \times A = \{(x,x,x), (x,x,y), (x,y,x), (x,y,y), (y,x,x), (y,x,y), (y,y,x), (y,y,y)\}$$

(b) (4 pts.) Verify (A ∪ B) – (A ∩ B) = (A – B) ∪ (B – A) for the sets A = {1, 2, 3, 4} and B = {3, 4, 5, 6}.

$(A \cup B) = \{1,2,3,4,5,6\}$ $\quad (A \cap B) = \{3,4\}$

$(A \cup B) - (A \cap B) = \{1,2,3,4,5,6\} - \{3,4\} = \{1,2,5,6\}$

=

$(A - B) = \{1,2\}$ $\quad (B - A) = \{5,6\}$

$(A - B) \cup (B - A) = \{1,2\} \cup \{5,6\} = \{1,2,5,6\}$

Thus $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ for these sets.

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3. (a) (3 pts.) Circle the **BEST** choice (F = Function I = One-to-one S = Onto B = Bijection) for the following:

F I S B $F(x) = x$ on the set {1, 2, 3, 4, 5}.

F I S B $F(x) = x^2$ on **R**.

F I S B $F(x) = \{(1, 2), (2, 3), (3, 1), (4, 2)\}$ on the set {1, 2, 3, 4}.

(b) (3 pts.) What is the inverse of {(1, 2), (2, 3), (3, 1), (4, 2)}?

$\{(2,1),(3,2),(1,3),(2,4)\}$

(c) (3 pts.) Fill in the blanks: $|\mathcal{P}(\{1, 2, 3, 4, 5\})| =$ 2^5 or 32 d(10110) = 3 H(11011, 01010) = 2.

$\oplus 01010$
10001

(d) (3 pts.) Give an example of a Bijective (1-1 and onto) Function on {1, 2, 3, 5, 8, 13} that includes (3, 8), (5, 2), and (8, 3).

$\{(3,8),(5,2),(8,3),(1,\overset{1}{\underset{13}{5}}),(2,\overset{1}{\underset{13}{5}}),(13,\overset{1}{\underset{13}{5}})\}$

4. (a) (4 pts.) Use the Euclidean Algorithm to find GCD(144, 84).

$144 = 84 + 60$
$84 = 60 + 24$ → $GCD(144,84) = GCD(84,60)$
$60 = 2(24) + 12$ $= GCD(60,24)$
$24 = 2(12) + 0$ $= GCD(24,12)$
$= GCD(12,0) = 12$

(b) (4 pts.) Show the algorithm whose complexity is $(n^7)(2n^4 + 2n^5\log n)(n^3\log n + n^2 + 4)$ is $O(n^{17})$.

$$n^7(2n^4+2n^5\log n)(n^3\log n+n^2+4) \le n^7[2n^4+2n^5(n)][n^3(n)+n^2+4]$$
$$= n^7(2n^4+2n^6)(n^4+n^2+4)$$
$$\le n^7(2n^6+2n^6)(n^4+n^4+4n^4)$$
$$= n^7(4n^6)(6n^4) = 24n^{17} \rightarrow O(n^{17})$$

5. (a) (3 pts.) Give an inductive definition for the set of powers of 3.

Basis: $3^0 = 1 \in$ {Powers of 3}

Induction: If $x \in$ {Powers of 3}, then $(3x) \in$ {Powers of 3}.

(b) (3 pts.) Find an expression void of summations for the series: $\sum_{i=0}^{n}(2^i + 4i)$.

$$\sum_{i=0}^{n}(2^i+4i) = \sum_{i=0}^{n}2^i + 4\sum_{i=0}^{n}i = (2^{n+1}-1)+4\left[\frac{n(n+1)}{2}\right]$$
$$= 2^{n+1}-1+2n^2+2n.$$

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6. (10 pts.) Prove 1 of the 2 Theorems below:

Theorem 1: If the sum of the digits of a 3-digit Natural is divisible by 3, then the Natural is divisible by 3. (i.e. 324 = 3(100) + 2(10) + 4 = 3(108) and 3 + 2 + 4 = 3(3))

Theorem 2: The Rationals are closed under subtraction.

Theorem 1: Let $a+b+c=3k$. Show $(100a+10b+c)=3p$ for some $p \in \mathbb{Z}$.

Now $(33a+3b) \in \mathbb{Z}$, so $100a+10b+c = (99a+9b)+(a+b+c)$

$= 3(33a+3b)+3k$

$= 3[(33a+3b)+k]$.

Since $p=[(33a+3b)+k] \in \mathbb{Z}$, we see that $100a+10b+c=3p$. QED

Theorem 2: For all $a,b,c,d \in \mathbb{Z}$ with $b,d \neq 0$, we have that

$\frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd}$. Since $b,d \neq 0$, $bd \neq 0$ and

$(ad-cb) \in \mathbb{Z}$ and $(bd) \in \mathbb{Z}$, therefore

$\left(a/b - c/d\right)$ is Rational. QED

7. (10 pts.) Prove 1 of the 2 Theorems that follow by Mathematical Induction.

Theorem 1: $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Theorem 2: If $s_n = (s_{n-1})(s_{n-2})(s_{n-3})$ with $s_0 = 1$, $s_1 = 5$, and $s_2 = 5^2$, then s_n is a power of 5, for all $n \geq 3$.

Theorem 1: Basis $\sum_{k=0}^{0} k^2 = 0^2 = 0 = \frac{0(1)(1)}{6}$.

Induction: Assume $\sum_{k=0}^{p} k^2 = \frac{p(p+1)(2p+1)}{6}$. Now $\sum_{k=0}^{p+1} k^2 = \sum_{k=0}^{p} k^2 + \sum_{k=p+1}^{p+1} k^2 = \frac{p(p+1)(2p+1)}{6} + (p+1)^2$

So $\sum_{k=0}^{p+1} k^2 = \frac{(p+1)}{6}[p(2p+1)+6(p+1)] = \frac{(p+1)}{6}[2p^2+p+6p+6] = \frac{(p+1)}{6}(2p^2+7p+6) = \frac{(p+1)(p+2)(2p+3)}{6}$ QED

Theorem 2: Basis: $S_3 = S_2 S_1 S_0 = 5^2 \cdot 5 \cdot 1 = 5^3$, so S_3 is a power of 5.

Induction: Let $S_4, S_5, \ldots, S_{p-1}$ each be a power of 5.

Now $S_p = S_{p-1} S_{p-2} S_{p-3} = 5^i 5^j 5^k$ for some $i,j,k \in \mathbb{Z}$.

Thus $S_p = 5^{(i+j+k)}$, therefore S_p is a power of 5. QED

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8. (a) (3 pts.) A restaurant serves 4 soups, 3 salads, 12 entrees, 4 desserts, and 9 beverages. How many dinners can they create if each dinner consists of a soup or salad, an entree, and a dessert or beverage?

$$(4+3)12(4+9) = 7\cdot 12\cdot 13$$

(b) (3 pts.) How many ways can I seat 12 students at a round table if a certain pair cannot be seated together?

all − (ways with AB) − (ways with BA)

$$(12-1)! - (11-1)! - (11-1)! = 11! - 2\cdot 10!$$

(c) (3 pts.) How many ways can judges award 1st, 2nd, 3rd, 4th, and 5th Place prizes to 100 contestants?

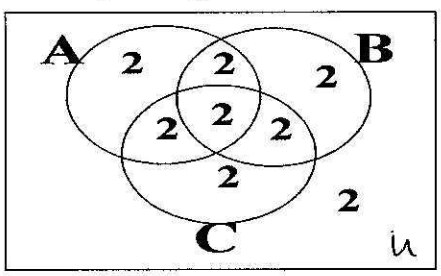
$$P(100,5) = \frac{100!}{(100-5)!} = \frac{100!}{95!} \text{ or } 100\cdot 99\cdot 98\cdot 97\cdot 96$$

(d) (3 pts.) How many distinct piles of 100 coins (pennies, nickels, dimes, quarters, half-dollars, and dollars) can I create from a vast quantity of coins, if I must have at least 12 of each in its pile?

$S=100 \quad C=6 \quad T=5 \quad R=6\cdot 12=72$

$$\binom{100+5-72}{100-72} = \binom{33}{28} = \frac{33!}{28!\,5!}$$

9. Consider the following sets with corresponding number of elements indicated in each region:



(a) (3 pts.) Find P(A ∪ C)

$$P(A\cup C) = \frac{|A\cup C|}{|u|} = \frac{(2+2+2+2+2+2)}{2\cdot 8} = 12/16 = 3/4$$

(b) (3 pts.) Find P((A ∩ C) | (B ∪ C))

$$P(A\cap C \mid B\cup C) = \frac{|(A\cap C)\cap(B\cup C)|}{|B\cup C|} = \frac{|A\cap C|}{|B\cup C|} = \frac{4}{6\cdot 2} = \frac{4}{12} = 1/3$$

(c) (3 pts.) Determine whether or not (A ∩ C) is Independent of (B ∪ C)

Independent if $P(A\cap C \mid B\cup C) = P(A\cap C)$. $P(A\cap C \mid B\cup C) = 1/3$ $P(A\cap C) = \frac{4}{16} = \frac{1}{4}$ and $1/3 \neq 1/4$, therefore <u>NOT</u> independent.

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10. (a) (3 pts.) Find $P_{1,3}$ for the database whose records are:

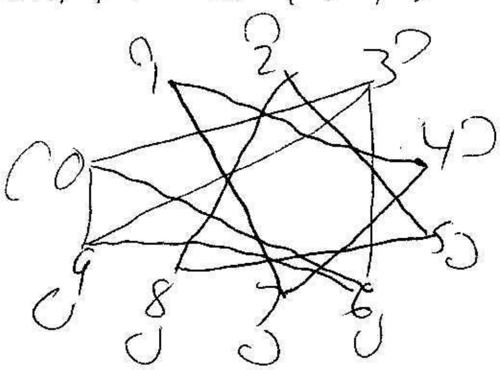
{(1, 2, a), (1, 3, a), (2, 3, b), (3, 1, a), (3, 2, c), (3, 3, b), (3, 4, a), (4, 1, c), (4, 2, c), (5, 2, a), (5, 4, b)}

$P_{1,3} = \{(1,a), \cancel{(1,a)}, (2,b), (3,a), (3,c), (3,b), \cancel{(3,a)}, (4,c), \cancel{(4,c)}, (5,a), (5,b)\}$

$= \{(1,a), (2,b), (3,a), (3,c), (3,b), (4,c), (5,a), (5,b)\}$

(b) (3 pts.) Graph $S = \{(a,b) \mid a,b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $a \equiv b \bmod 3\}$.

$[0] = \{0,3,6,9\}$ $[1] = \{1,4,7\}$ $[2] = \{2,5,8\}$



(c) (3 pts.) What partition of {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} does S induce?

$\{0,3,6,9\} \cup \{1,4,7\} \cup \{2,5,8\}$

11. (8 pts.) Find the Disjunctive Normal Form for the function of a 3-way switch controlling a light bulb so that the bulb is ON when all 3 switches are ON.

x	y	z	F(x,y,z)	
1	1	1	1	$\to xyz$
1	1	0	0	
1	0	1	0	
1	0	0	1	$\to xy'z'$
0	1	1	0	
0	1	0	1	$\to x'yz'$
0	0	1	1	$\to x'y'z$
0	0	0	0	

$F(x,y,z) = xyz + xy'z' + x'yz' + x'y'z$