CMSC 203 - Discrete Structures - FINAL EXAM - Fall 2009 **Name** ANSWER KEY

1. (a) (3 pts.) Use the Laws of Logic to show: $(\neg p \to r) \wedge (\neg q \to r) \equiv \neg(p \wedge q) \to r$

$$
\begin{aligned}
(\neg p \to r) \wedge (\neg q \to r) &\equiv (\neg(\neg p) \vee r) \wedge (\neg(\neg q) \vee r) \\
&\equiv (p \vee r) \wedge (q \vee r) \\
&\equiv (p \wedge q) \vee r \\
&\equiv \neg(\neg(p \wedge q)) \vee r \\
&\equiv \neg(p \wedge q) \to r.
\end{aligned}
$$

(b) (3 pts.) Find the negation of the statement: **All, who run fast, win races.**

Some run fast and do not win races.

(c) (3 pts.) Use the rules of inference to show the following is a valid argument:

$p \to q$
$r \to w$
$\neg q$
$r \vee s$
$s \to p$
$\therefore w$

(1) $\dfrac{p \to q \quad \neg q}{\therefore \neg p}$ $\quad \dfrac{s \to p \quad \neg p}{\therefore \neg s}$ $\quad \dfrac{r \vee s \quad \neg s}{\therefore r}$ $\quad \dfrac{r \to w \quad r}{\therefore w}$ QED

2. (a) (3 pts.) Find A × B for the sets A = {10, 11, 01} and B = {100, 010, 001}.

$A \times B = \{ (10, 100), (10, 010), (10, 001),$
$(11, 100), (11, 010), (11, 001),$
$(01, 100), (01, 010), (01, 001) \}$

(b) (4 pts.) Use the Properties of Sets, to show $(A \cup B) - (A \cap B) = (A - B) \cap (B - A)$.

$$
\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c) \\
&= [(A \cup B) \cap A^c] \cup [(A \cup B) \cap B^c] \\
&= [(A \cap A^c) \cup (B \cap A^c)] \cup [(A \cap B^c) \cup (B \cap B^c)] \\
&= [\emptyset \cup (B - A)] \cup [(A - B) \cup \emptyset] \\
&= (B - A) \cup (A - B) \\
&= (A - B) \cup (B - A)
\end{aligned}
$$

CMSC 203 - Discrete Structures - FINAL EXAM - Fall 2009 **Name** ANSWER KEY

3. (a) (3 pts.) What Domain, Image and Range make F = {(3, 2), (1, 5), (5, 4), (6, 3), (7, 9), (8, 8)} a One-to-One Function:

DOMAIN: {3, 1, 5, 6, 7, 8}

IMAGE: {2, 5, 4, 3, 9, 8}

RANGE: Anything containing {2, 5, 4, 3, 9, 8}

(b) (3 pts.) What is the inverse of F above?

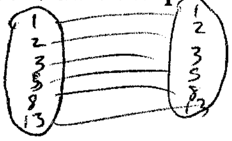
$F^{-1} = \{(2,3),(5,1),(4,5),(3,6),(9,7),(8,8)\}$

(c) (3 pts.) H(01101101, 11011010) = 6.

11011010

10110111

(d) (3 pts.) Give an example of a Bijective (1-1 and onto) Function on {1, 2, 3, 5, 8, 13}



4. (a) (4 pts.) Use the Euclidean Algorithm to find GCD(148, 70).

$148 = 70(2) + 8 \Rightarrow GCD(148, 70) = GCD(70, 8) = GCD(8, 6) = GCD(6, 2)$

$70 = 8(8) + 6$ $\quad = GCD(2, 0) = 2.$

$8 = 6(1) + 2$

$6 = 2(3) + 0$

(b) (4 pts.) Show the algorithm whose complexity is $(n^6)(2n^7 + 2n^5)(3n^4 + 3n^2 + 1)$ is $O(n^{17})$.

$$n^6(2n^7+2n^5)(3n^4+3n^2+1) \le n^6(2n^7+2n^7)(3n^4+3n^4+n^4)$$
$$= n^6(4n^7)(7n^4)$$
$$= 28n^{17} \Rightarrow O(n^{17}).$$

5. (a) (3 pts.) Rewrite 2 – 8 + 32 – 128 + 512 – 2048 + 8192 as a summation from 1 to 7.

$$2^1 - (2)^3 + (2)^5 - (2)^7 + (2)^9 - (2)^{11} + (2)^{13}$$
$$= \sum_{i=1}^{7} (-1)^{(i+1)} (2)^{(2i-1)} \quad \text{or} \quad \sum_{i=1}^{7} (-1)^{(i-1)} 2^{(2i-1)}$$

(b) (3 pts.) Find an expression void of summations for the series: $\sum_{i=0}^{n} (3^i + 2i)$.

$$\sum_{i=0}^{n} (3^i + 2i) = \sum_{i=0}^{n} 3^i + \sum_{i=0}^{n} 2i = \sum_{i=0}^{n} 3^i + 2\sum_{i=0}^{n} i$$
$$= \frac{3^{(n+1)} - 1}{3-1} + 2\left[\frac{n(n+1)}{2}\right] = \frac{3^{(n+1)} - 1}{2} + n(n+1)$$
$$\text{or } \frac{3^{(n+1)} + 2n^2 + 2n - 1}{2}$$

6. (10 pts.) Prove 1 of the 2 Theorems below:
Theorem 1: If a prime number divides the square of an integer, then it divides the integer.
Theorem 2: The square root of a prime integer is irrational. (Assume Theorem 1)

Theorem 1 Proof (Contradiction) Assume p is prime and $p \mid n^2$ but $p \nmid n$.
By the QRT, $\exists\, q, r \in \mathbb{Z} \ni n = pq + r$ and since $p \nmid n$ $0 < r < p$.
Now $n^2 = (pq+r)^2 = (p^2q^2 + 2pqr + r^2)$ and $p/n \Rightarrow p \mid r^2$,
hence $r^2 = kp$

Directly w/ $n = p_1 p_2 \cdots p_k \Rightarrow n^2 = p_1^2 p_2^2 \cdots p_k^2$ but p/n^2
$\Rightarrow p$ is one of $p_1, p_2, \ldots, p_k$. Therefore p/n.

7. (10 pts.) Prove 1 of the 2 Theorems that follow by Mathematical Induction.
Theorem 1: There are 2^n binary strings of length n.
Theorem 2: If $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ when $s_0 = 11$, $s_1 = 33$, and $s_2 = 55$, then s_n is odd, for all $n \geq 3$.

CMSC 203 - Discrete Structures - FINAL EXAM - Fall 2009 **Name** Answer Key

8. (a) (3 pts.) A restaurant serves 6 soups, 4 salads, 14 entrees, 8 desserts, and 9 beverages. How many dinners can they create if each dinner consists of a soup or salad, an entree, and a dessert or beverage?

$(6+4)(14)(8+9) = 10(14)(17).$

(b) (3 pts.) How many ways can I choose 5 students from a class of 30 if a certain pair cannot be selected together?

All − choices w/ the pair $= C(30,5) - C(28,3).$

(c) (3 pts.) How many ways can judges award 1st, 2nd, 3rd, and 4th Place prizes to 50 contestants?

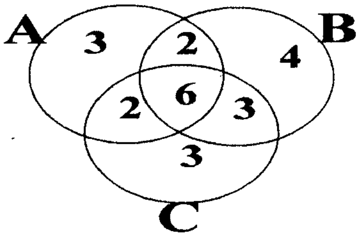
$P(50,4) = \frac{50!}{(50-4)!} = \frac{50!}{46!}$ or $50 \cdot 49 \cdot 48 \cdot 47$

(d) (3 pts.) How many distinct piles of 300 coins (pennies, nickels, dimes, quarters, half-dollars, and dollars) can I create from a vast quantity of coins, if I must have at least 10 of each in its pile?

slots = 300 cats = 6 rest = 6·10 = 60

$\binom{300+6-1-60}{300-60} = \binom{245}{240} = \binom{245}{5}$

9. (3 pts.) Consider the following sets with corresponding number of elements indicated in each region:



$|A \cup C| = 3+2+2+6+3+3 = 19$

$|S| = 3+2+4+2+6+3+3 = 23$

$|(A \cap C) \cap (B \cup C)| = 6$

$|B \cup C| = 2+4+6+3+2+3 = 20$

$|A \cap C| = 2+6 = 8$

(a) (3 pts.) Find P(A ∪ C)

$P(A \cup C) = \frac{|A \cup C|}{|S|} = \frac{19}{23}$

(b) (3 pts.) Find P((A ∩ C) | (B ∪ C))

$P((A \cap C) \mid (B \cup C)) = \frac{|(A \cap C) \cap (B \cup C)|}{|B \cup C|} = 8/20 = 4/10 = 2/5$

(c) (3 pts.) Determine whether or not (A ∩ C) is Independent of (B ∪ C) $\Rightarrow$ NO.

Is $P((A \cap C) \cap (B \cup C)) = P(A \cap C)P(B \cup C)$?

$= 8/23$

$= 8/23 \cdot 20/23 = \frac{160}{(23)^2} \neq \frac{184}{(23)^2} = \frac{8(23)}{23^2} = \frac{8}{23}.$

CMSC 203 - Discrete Structures - FINAL EXAM - Fall 2009 Name ANSWER KEY

10. (a) (3 pts.) Find $P_{2,3}$ for the database whose records are:
{(1, 2, a), (1, 3, a), (2, 3, b), (3, 1, a), (3, 2, c), (3, 3, b), (3, 4, a), (4, 1, c), (4, 2, c), (5, 2, a), (5, 4, b)}

$P_{2,3} = \{(2,a),(3,a),(3,b),(1,a),(2,c),(4,a),(1,c),(4,b)\}$

$(1,ac),(2,ac),(3,ab),(4,ab)$

(b) (3 pts.) Show that the relation S = {(*a*,*b*) | *a*,*b* are Real and |*a*| + 1 = |*b*| + 1} is an Equivalence Relation.

$\forall a \in \mathbb{R},\ |a|+1 = |a|+1 \Rightarrow (a,a) \in S \Rightarrow$ REFL.

If $(a,b) \in S \Rightarrow |a|+1 = |b|+1 \Rightarrow |b|+1 = |a|+1 \Rightarrow (b,a) \in S \Rightarrow$ SYMM.

If $(a,b) \in S$ and $(b,c) \in S \Rightarrow |a|+1 = |b|+1 = |c|+1 \Rightarrow |a|+1 = |c|+1$

$\Rightarrow (a,c) \in S \Rightarrow$ TRANS.

(c) (3 pts.) What partition of the Reals does S induce?

$\bigcup_{s \in \mathbb{R}_+} [s]$ where $[s] = \{s, -s\}$.

11. (8 pts.) Find the Truth Table and a Normal Form (either Disjunctive or Conjunctive) of F(*x*, *y*, *z*) = *xy* + *z*'

$F(x,y,z) = xy + z' = xy(z+z') + (x+x')(y+y')z' = xyz + xyz' + xy'z' + x'yz' + x'y'z'$

$= (x'+y+z')(x+y'+z')(x+y+z')$

x	y	z	F(x,y,z)
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

$(xy+z') = (x+z')(y+z')$

$= (x+z'+yy')(xx'+y+z')$

$= \cancel{(x+y+z')}(x+y'+z')(x+y+z')(x'+y+z')$ ✓