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Show All Work!

1. (16 points) Circle **T** if the corresponding statement is True or **F** if it is False.

T (**F**) GCD(10, 0) = 0.

(**T**) **F** If a prime divides the square of a Natural number, then it divides the number.

T (**F**) If $a \equiv b$ MOD 5, then 5 divides both a and b.

(**T**) **F** Recursive algorithms generally use memory more efficiently than their equivalent Iterative version.

(**T**) **F** 1 + 2 + 3 + 4 + ... + 1000 = 500(1001).

T (**F**) Algorithms with $O(n^2)$ are less efficient than those with $O(2^n)$.

(**T**) **F** The two Principles of Mathematical Induction are logically equivalent.

(**T**) **F** GCD(1000, 678) = GCD(678, 322).

2. (8 points) Find the GCD and LCM of $a = 2^4 5^8 7^3 11^1 17^6$ and $b = 2^3 5^4 11^2 13^2 17^2 19^2$

$ab = 2^{4+3} 5^{8+4} 7^{3+0} 11^{1+2} 13^{0+2} 17^{6+2} 19^{0+2}$

so $GCD(a,b) = 2^3 5^4 7^0 11^1 13^0 17^2 19^0 = 2^3 5^4 11^1 17^2$

and $LCM(a,b) = 2^4 5^8 7^3 11^2 13^2 17^6 19^2$.

3. (8 points) List out the search intervals of the Binary Search algorithm to find 6 in the list:

3 4 6 9 13 18 21 34 55 72 83 85 92 104 111 133

Pass 1: {3, 4, 6, 9, 13, 18, 21, 34} and {55, 72, 83, 85, 92, 104, 111, 133}

Pass 2: {3, 4, 6, 9} and {13, 18, 21, 34}

Pass 3: {3, 4} and {6, 9}

Pass 4: {6} and {9}

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4. (10 points) Find a numeric expression for $\sum_{i=0}^{10} 4i + 5(7^i)$.

$$\sum_{i=0}^{10} 4i + 5(7^i) \quad = \quad 4\sum_{i=0}^{10} i + 5\sum_{i=0}^{10} 7^i \quad = \quad 4\frac{10(11)}{2} + 5\frac{7^{10+1} - 1}{7 - 1} \quad = \quad 220 + \frac{5(7^{11} - 1)}{6}$$

5. (12 points) Trace the Division Algorithm below to find (52 MOD 6).

```
PROCEDURE MOD(A,B: integers)
WHILE (A > B)
  A = A – B
ENDWHILE
OUTPUT (A)
```

Step	0	1	2	3	4	5	6	7	8
A	52	46	40	34	28	22	16	10	4
B	6	6	6	6	6	6	6	6	6
(A > B)?	1	1	1	1	1	1	1	1	0
OUTPUT									4.

6. (8 points) Give a Recursive Definition for the set $S = \{n \in \mathbf{N} \mid n \equiv 3 \text{ MOD } 7\}$:

Basis: $3 \in S$

Induction: If $n \in S$, then $(n + 7) \in S$.

7. (8 points) Show n^{19} is the Big-Oh of the algorithm with complexity:

$$(12n^4 + 3n^3\log^3 n)(3n^7 + 4n^3)(n^6 + 5n^2 + 4).$$

$$(12n^4 + 3n^3\log^3 n)(3n^7 + 4n^3)(n^6 + 5n^2 + 4) \le (12n^4 + 3n^3n^3)(3n^7 + 4n^7)(n^6 + 5n^6 + 4n^6)$$
$$\le (12n^6 + 3n^6)(7n^7)(10n^6) = (15n^6)(7n^7)(10n^6) = 1050n^{19} \text{ which is } O(n^{19}).$$

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8. (10 points) Prove ONE of the TWO Theorems below using Mathematical Induction.

Theorem 1: For all Natural numbers n, $\sum_{i=0}^{n} 7^i = \frac{7^{n+1} - 1}{6}$.

Theorem 2: If $a_0 = 1$, $a_1 = 10$, $a_2 = 100$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, then $a_n \leq 10^n$, for all $n \geq 3$.

Theorem 1: Proof (Weak/First Induction):

Basis: Show true for $n = 0$. Now, $\sum_{i=0}^{0} 7^i = 7^0 = 1$ and $\frac{7^{0+1} - 1}{6} = \frac{7^1 - 1}{6} = \frac{6}{6} = 1$, thus

$\sum_{i=0}^{n} 7^i = \frac{7^{n+1} - 1}{6}$ for $n = 0$.

Induction: Assume true for $n = k$ and show true for $n = (k + 1)$. Assume $\sum_{i=0}^{k} 7^i = \frac{7^{k+1} - 1}{6}$.

Now, $\sum_{i=0}^{k+1} 7^i = \sum_{i=0}^{k} 7^i + \sum_{i=k+1}^{k+1} 7^i = \frac{7^{k+1} - 1}{6} + 7^{k+1} = \frac{7^{k+1} - 1 + 6(7^{k+1})}{6} = \frac{7(7^{k+1}) - 1}{6} = \frac{7^{k+2} - 1}{6}$.

Since, $\frac{7^{k+2} - 1}{6} = \frac{7^{(k+1)+1} - 1}{6}$, we see that $\sum_{i=0}^{k+1} 7^i = \frac{7^{(k+1)+1} - 1}{6}$, therefore

$\sum_{i=0}^{n} 7^i = \frac{7^{n+1} - 1}{6}$ for all Natural Numbers, n. QED

Theorem 2: Proof: (Strong/Second Induction)

Basis: Show true for $n = 3$. Now $a_3 = a_2 + a_1 + a_0 = 1 + 10 + 100 = 111 \leq 1000 = 10^3$,

hence $a_3 \leq 10^3$.

Induction: Assume $a_3 \leq 10^3$, $a_4 \leq 10^4$, $a_5 \leq 10^5$..., $a_k \leq 10^k$, for some $k > 3$. Show $a_{k+1} \leq 10^{k+1}$.

Now, $a_{k+1} = a_k + a_{k-1} + a_{k-2} \leq 10^k + 10^{k-1} + 10^{k-2} = (10^2 + 10 + 1)10^{k-2} = 111(10^{k-2}) \leq 1000(10^{k-2})$,

thus $a_{k+1} \leq 1000(10^{k-2}) = 10^3(10^{k-2}) = 10^{k+1}$.

Therefore $a_n \leq 10^n$, for all $n \geq 3$. QED

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Show All Work!

9. (10 points) Prove ONE of the TWO Theorems below:

Theorem 1: For all Integers, n, if n is odd, then $n^2 \equiv 1$ MOD 8.
(Hint: If an Integer is odd, then its successor is even.)

Theorem 2: Between any two distinct Real Numbers is another Real Number.

Theorem 1:

Proof: Let n be an odd Integer, so $n = 2k + 1$, for some Integer k. We want to show $n^2 \equiv 1$ MOD 8; that is $(n^2 - 1) = 8p$, for some Integer p.

Now, $(n^2 - 1) = (2k + 1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4k(k + 1)$. However, since k is an odd Integer, we see that $(k + 1)$, the successor of k, is even. This lets us assert that $(k + 1) = 2m$, for some Integer m.

Combining all this, we see that $(n^2 - 1) = 4k(k + 1) = 4k(2m) = 8km$. Moreover, since k and m are Integers, we conclude that $p = km$ is an Integer, thus $(n^2 - 1) = 8p$, for some Integer p.

Therefore $n^2 \equiv 1$ MOD 8 for any odd Integer n. QED

Theorem 2:

Proof: Let X and Y be distince Real Numbers and, without loss of generality, assume X < Y.

Now, $X = (2X) / 2 = (X + X) / 2 < (X + Y) / 2 < (Y + Y) / 2 = (2Y) / 2 = Y$. Moreover, since X and Y are Real, we see that (X + Y) is Real, hence [(X + Y) / 2] is Real. Since $X < (X + Y) / 2 < Y$, we conclude, therefore, that there exists a Real Number between distinct Real Numbers. QED

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Show All Work!

10. (10 points) Prove ONE of the TWO Theorems below by Contradiction or Contraposition.

Theorem 1: The set of Prime Numbers is infinite.

Theorem 2: For all Integers, $n > 2$, if n is prime, then $n \equiv 1$ MOD 2.

Theorem 1: Proof: (Contradiction) Assume the set of Prime Numbers is finite. Denote the finite set of the Primes as $\{p_1, p_2, p_3, \dots , p_n\}$ for some Natural Number n.

Now, construct the Natural Number, $M = [p_1(p_2)(p_3)(\dots)(p_n)] + 1$. Since M is a Natural, it has a prime factor, but since any prime number is also a factor of the product $[p_1(p_2)(p_3)(\dots)(p_n)]$, we conclude that this prime factor must also divide 1. However this is a contradiction since no primes divide 1.

Therefore, the set of Primes is infinite. QED

Theorem 2: Proof: (Contraposition) We shall show for any Integer $n > 2$, if $n \equiv 0$ MOD 2, then n is composite.

Now, let $n \equiv 0$ MOD 2, for any Integer $n > 2$. This means that 2 divides $(n - 0) = n$, hence $n/2$ is an Integer. Moreover, since $n > 2$, we see that $n/2 > 1$, thus $n/2 = m$ for some Integer m. Consequently, $n = 2m$ with $m > 1$, hence n is composite.

Therefore, for all Integers, $n > 2$, if n is prime, then $n \equiv 1$ MOD 2. QED