Transformation Exercises: Denavit-Hartenberg Method

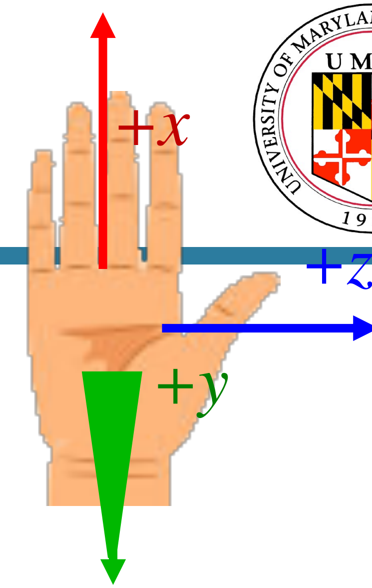
Some images and exercises from:
Introduction to Autonomous Mobile Robots, Siegwart, Nourbakhsh, 2011
Robot Dynamics and Control Second Edition, Spong, Hutchinson, Vidyasagar, 2004
Spacecraft Robot Kinematics Using Dual Quaternions, Valverde, Alfredo & Tsiotras, Panagiotis, 2018

Review

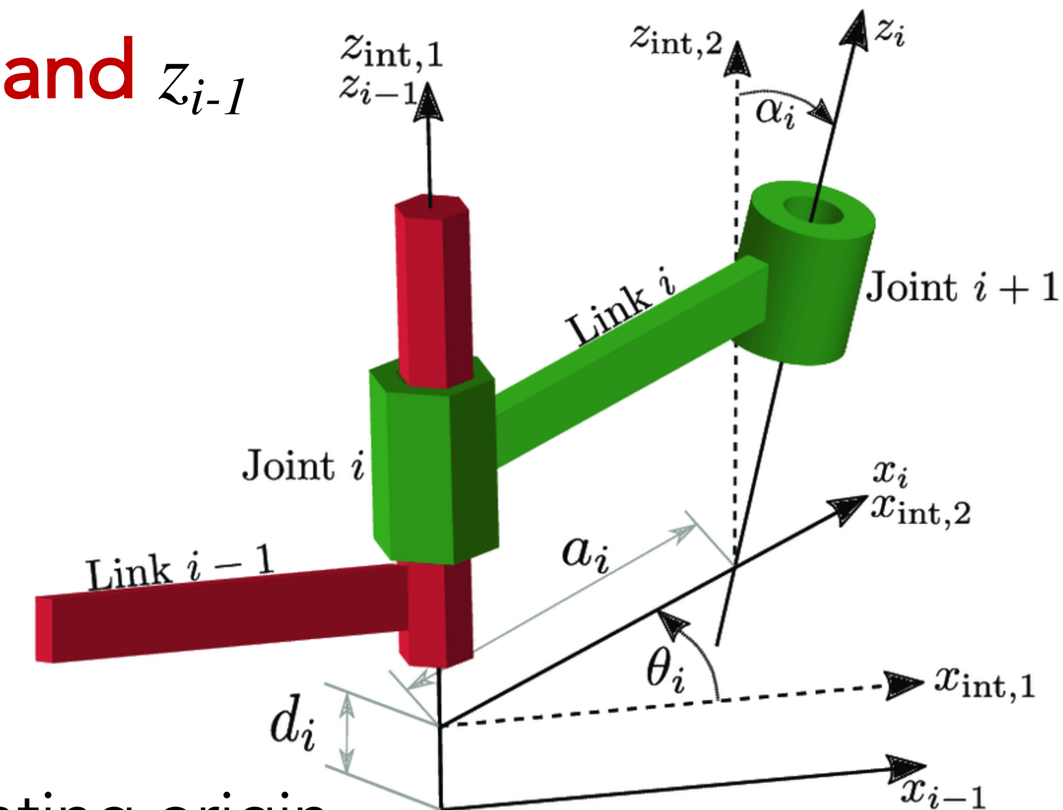
- Given two frames of reference, **transformations** convert configurations (position + orientation) from one to the other.
  - A robot sees a thing. Where is the thing in the world?
  - There's a thing in the world. Where is it wrt. the robot?
  - A robot moves around. Where is it in the world?
- We can do this with translation/rotation matrices, multiplied by configuration.
  - $\xi_W = {}^{W}_{R}T \times \xi_R$
- Or, faster with the Denavit-Hartenberg method.

https://matrix.reshish.com/multiplication.php is nice for running matrix multiplications

DH review: frames

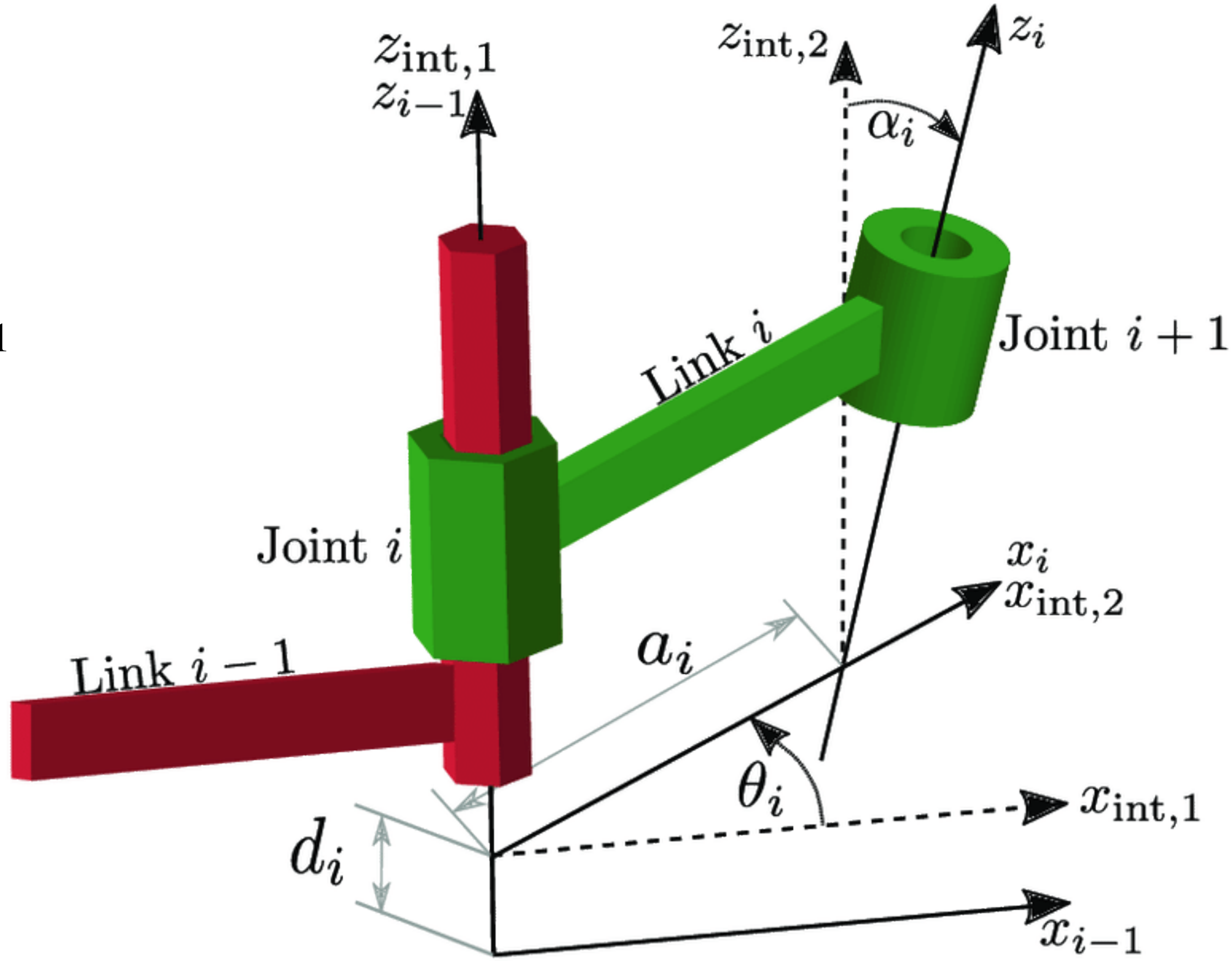


- z axis is axis of motion
  - Rotation around z for revolute
  - Translation along z for prismatic
- x_i axis orthogonal to z_i **and** z_{i-1}
- y axis: right-hand rule
  - Fingers point $+x$
  - Thumb points $+z$
  - Palm faces $+y$
- x_i axis must intersect z_{i-1} axis
  - Which may mean translating origin



DH review: parameters

- **a_{i-1} : link length**
  distance $Z_{i-1} \leftrightarrows Z_i$ along X_i
- **α_{i-1} : link twist**
  angle between $Z_{i-1} \leftrightarrows Z_i$ around X_i
- **d_i : link offset**
  distance X_{i-1} to X_i along Z_i
- **θ_i : joint angle**
  angle between X_{i-1} and X_i around Z_i



Valverde, Alfredo & Tsiotras, Panagiotis. (2018). Spacecraft Robot Kinematics Using Dual Quaternions. https://www.mdpi.com/2218-6581/7/4/64/htm

Review: Transformation matrices

$$T_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & cos\theta & -sin\theta & y \\ 0 & sin\theta & cos\theta & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

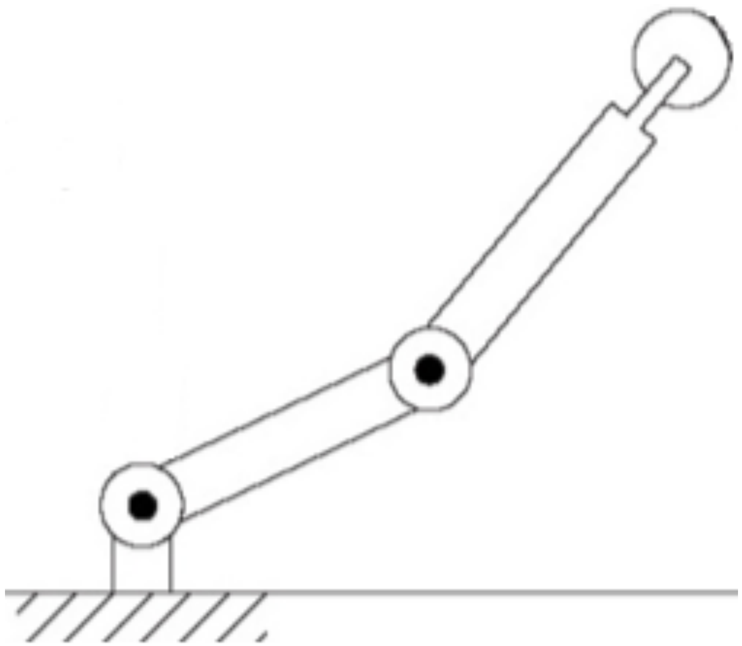
$$T_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta & x \\ 0 & 1 & 0 & y \\ -sin\theta & 0 & cos\theta & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 & x \\ sin\theta & cos\theta & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_{i,i+1} & \sin\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_{i,i+1} & -\cos\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\sin\theta_i \\ 0 & \sin\alpha_{i,i+1} & \cos\alpha_{i,i+1} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex.1: Planar elbow manipulatoı

- Define the axes according to the DH rules.

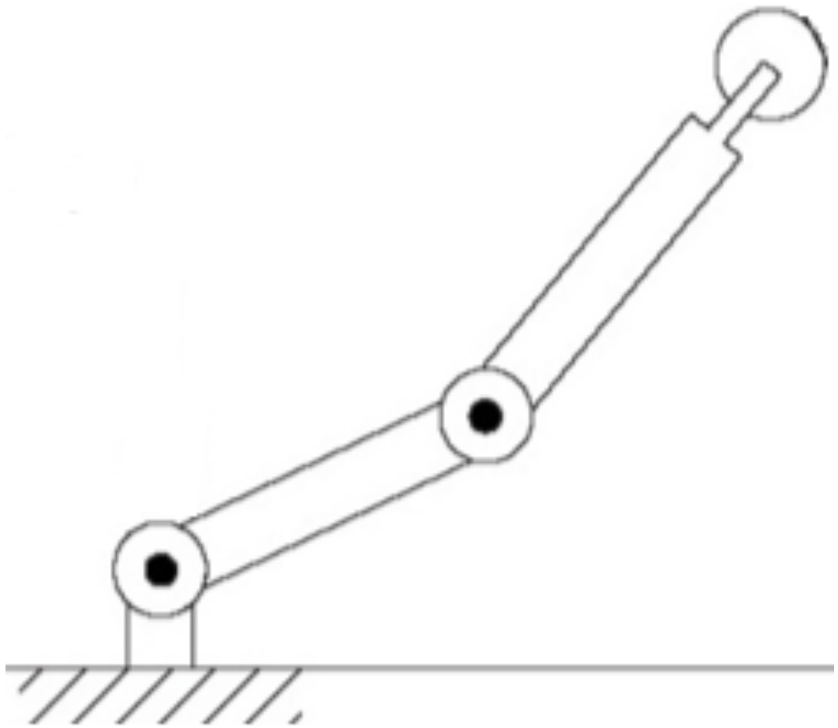


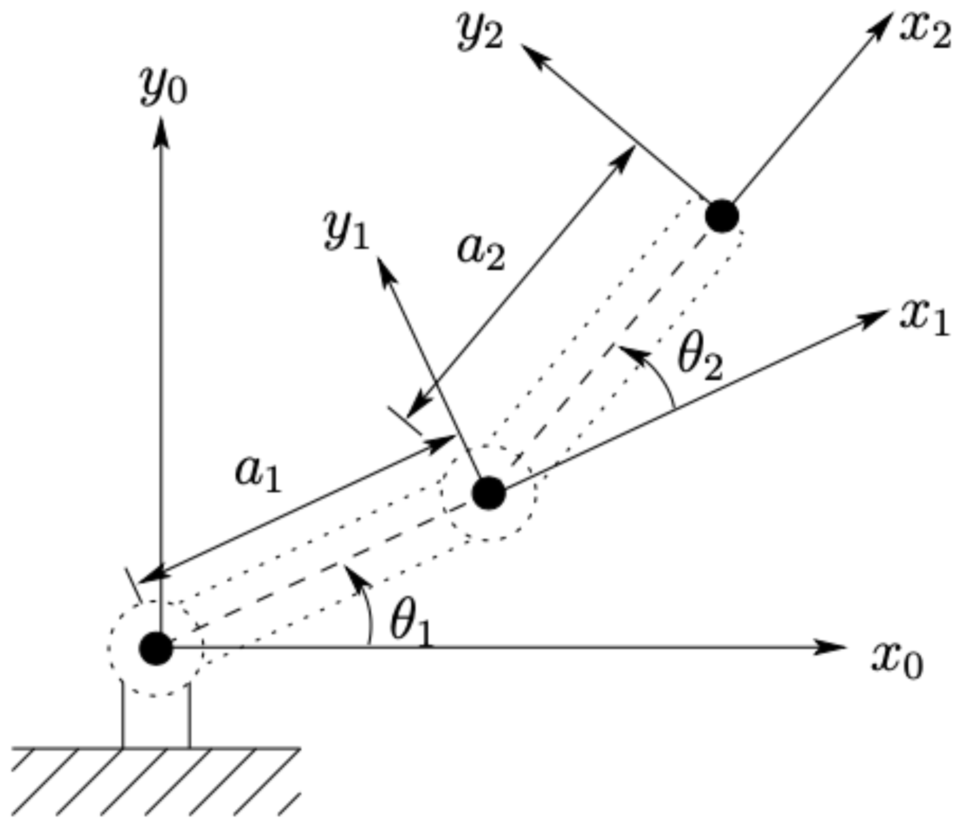
How many frames of reference do you need?

One per joint, so 2.

- Then, draw lines and arcs for a and θ.
  - Why not α and d?

Exercise 1

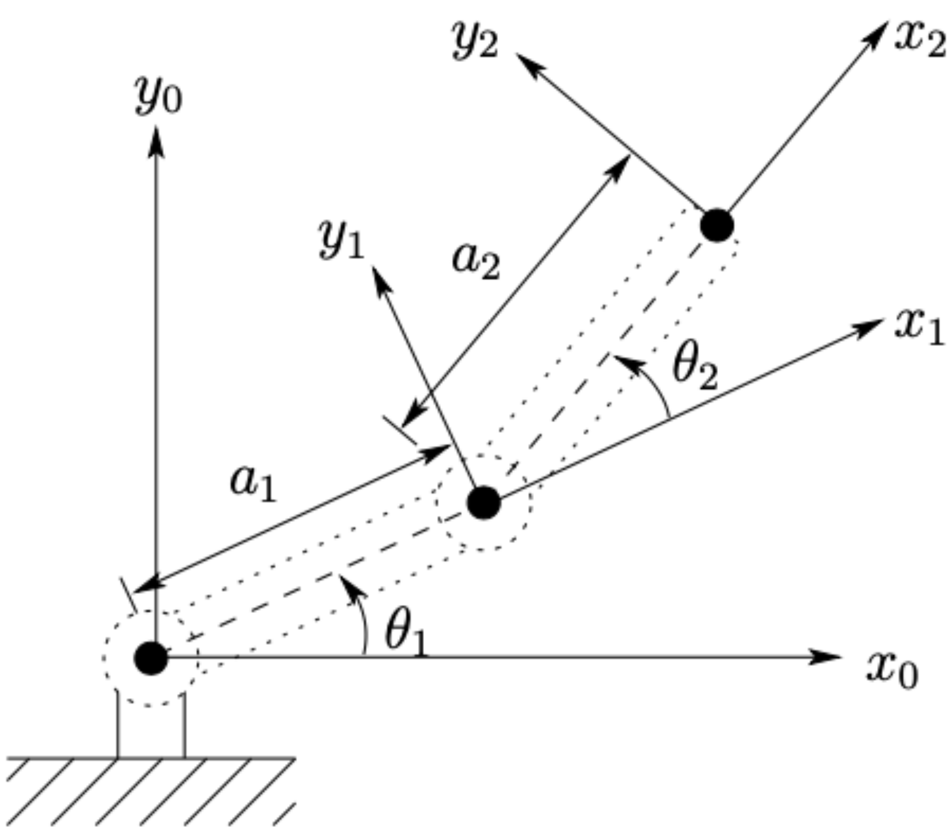




Z axes point out towards us

Exercise 1

- Give the DH parameters.



Z axes point out towards us

First, what's the table?

Link	a_i	α_i	d_i	θ_i

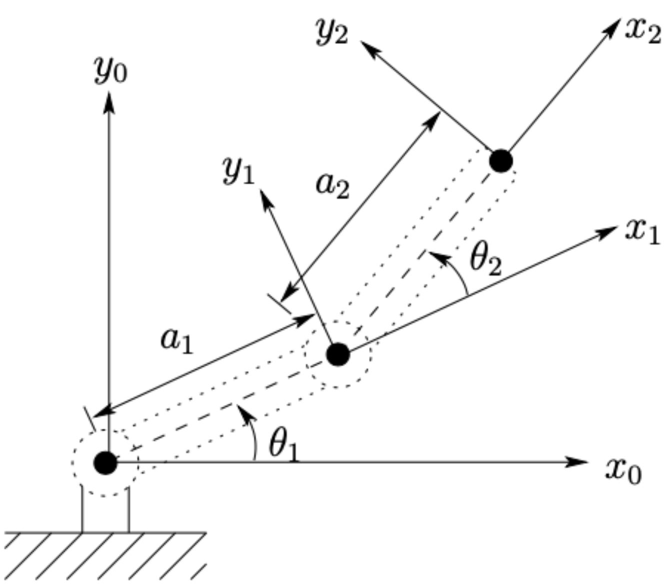
What are the values?

Planar arm makes params simpler, so…

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Exercise 1

- Give the final transformation matrix.



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

$$R_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_{i,i+1} & \sin\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_{i,i+1} & -\cos\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\sin\theta_i \\ 0 & \sin\alpha_{i,i+1} & \cos\alpha_{i,i+1} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 1

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- Plugging in…

$$T_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_{0,1}\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_{0,1}\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_{1,2}\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_{1,2}\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 1

- Give the final transformation matrix.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

$$R_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_{i,i+1} & \sin\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_{i,i+1} & -\cos\theta_i \sin\alpha_{i,i+1} & a_{i,i+1}\sin\theta_i \\ 0 & \sin\alpha_{i,i+1} & \cos\alpha_{i,i+1} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Plugging in…

$$T_2^0 = T_1^0 \times T_2^1 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_{0,1}\cos\theta_1 + a_{1,2}\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_{0,1}\sin\theta_1 + a_{1,2}\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 1

- This is **always** the parameters and transform for a planar elbow manipulator.

$$\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_{0,1}\cos\theta_1 + a_{1,2}\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_{0,1}\sin\theta_1 + a_{1,2}\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

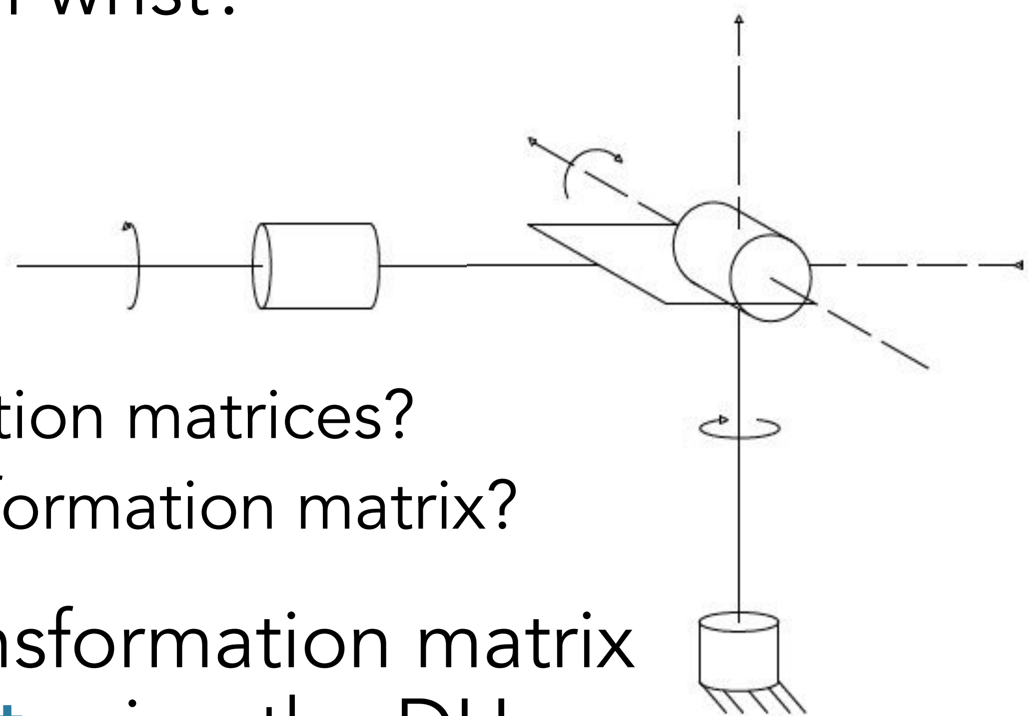
- Similar derivations can be done for cylindrical arms, spherical arms, etc.
  - This is why we name configurations.
- If you know these, you can subdivide an arm.

Discussion

- Could you do the same thing using a sequence of x/y/z rotations and translations?
  - How many steps would it take?
- Could you do the same thing given a word problem about an arm in the world?
  - Given numbers?
  - As a derivation?
- How many joints would you max out at?
  - Directly?
  - Using DH parameters?

Exercise 2:

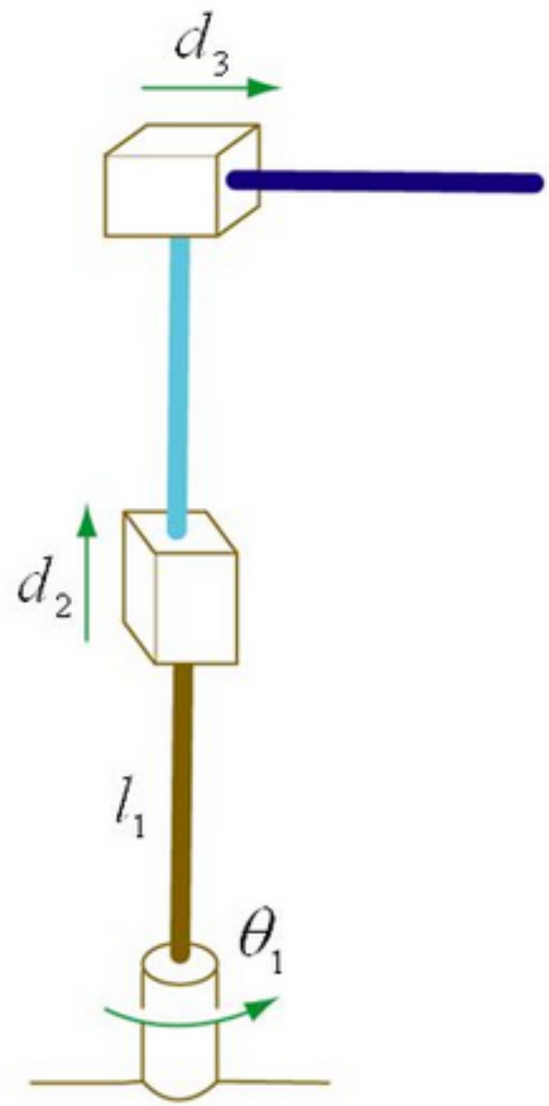
1. What is the complete, derived transformation matrix for a spherical wrist?
   a. What are the frames?
   b. What are the DH parameters?
   c. What are the individual transformation matrices?
   d. What's the final transformation matrix?
2. What is the final transformation matrix for this wrist, **without** using the DH method? (It's the same)



- This is just a sequence of rotations – see Spong.

Exercise 3:

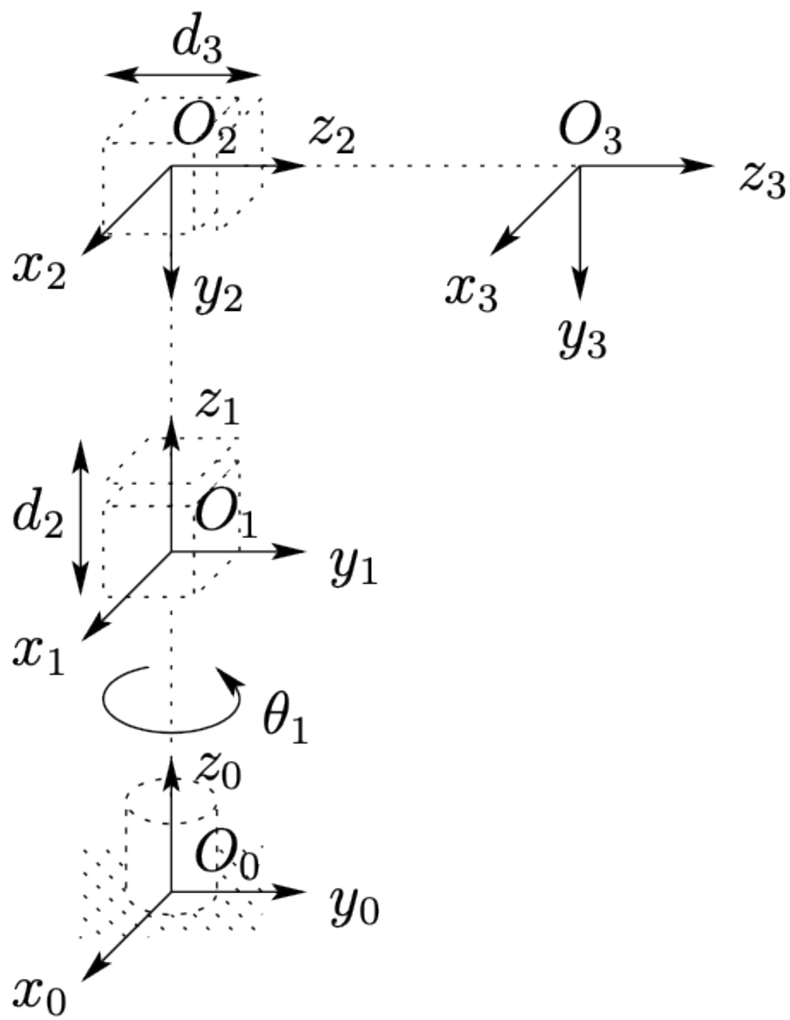
- What is the complete, derived transformation matrix for a 3-link cylindrical robot?
  a. What are the frames?
  b. What are the DH parameters?
  c. What are the individual transformation matrices?
  d. What's the final transformation matrix?
  e. What is the final transformation matrix for this wrist, derived **without** using the DH method?

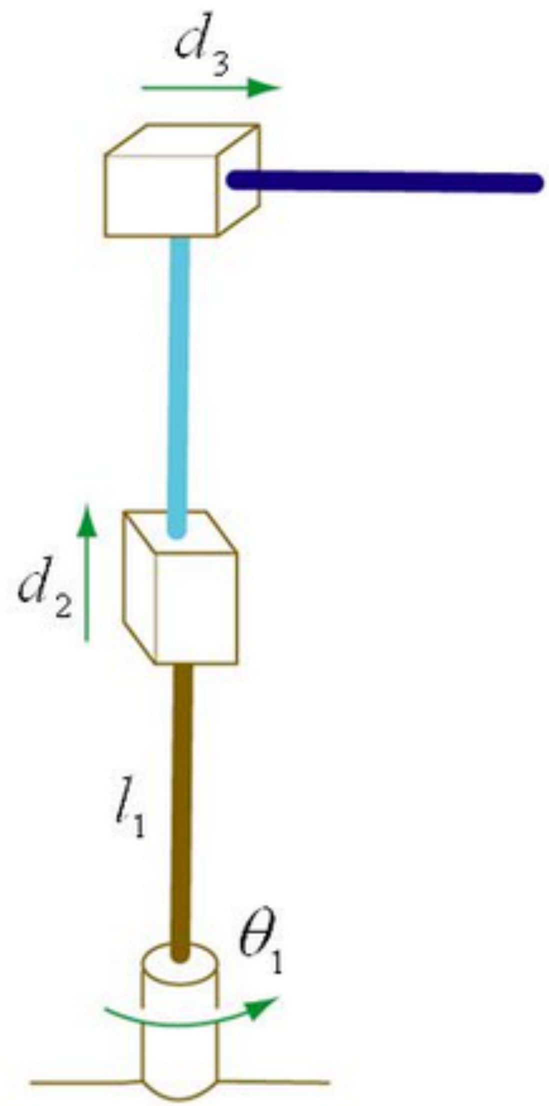


- See Spong.

Exercise 3:

a. What are the frames?

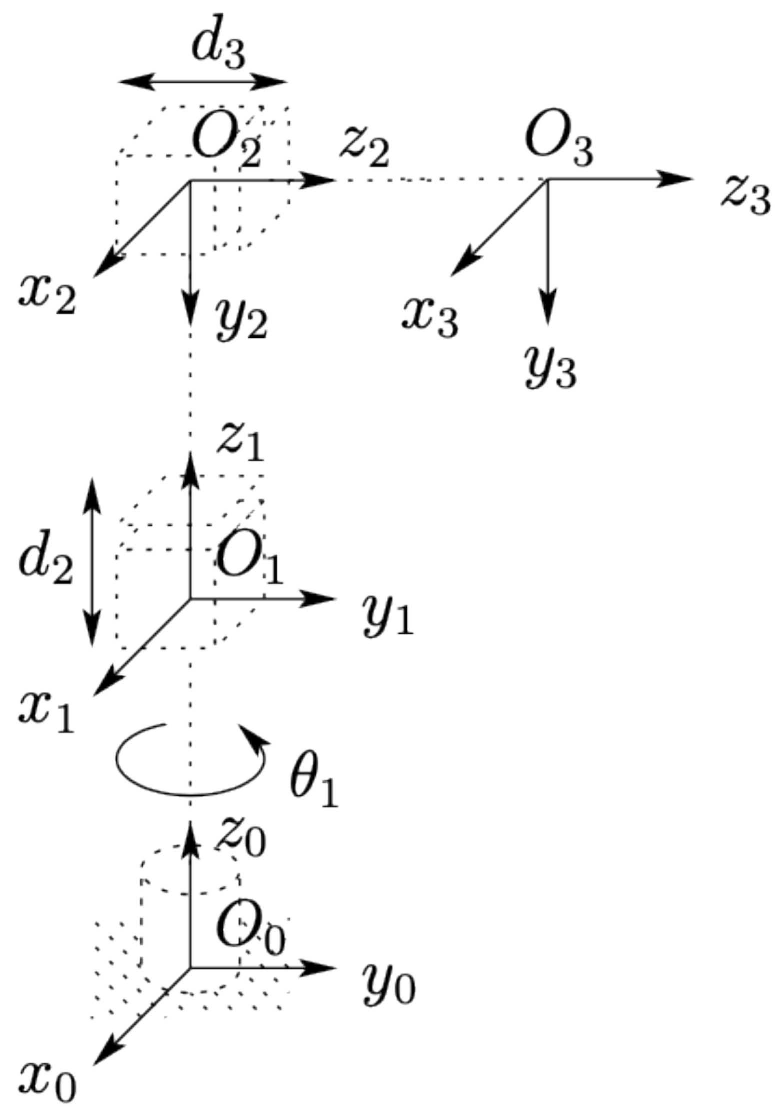




Exercise 3:

a. What are the frames?

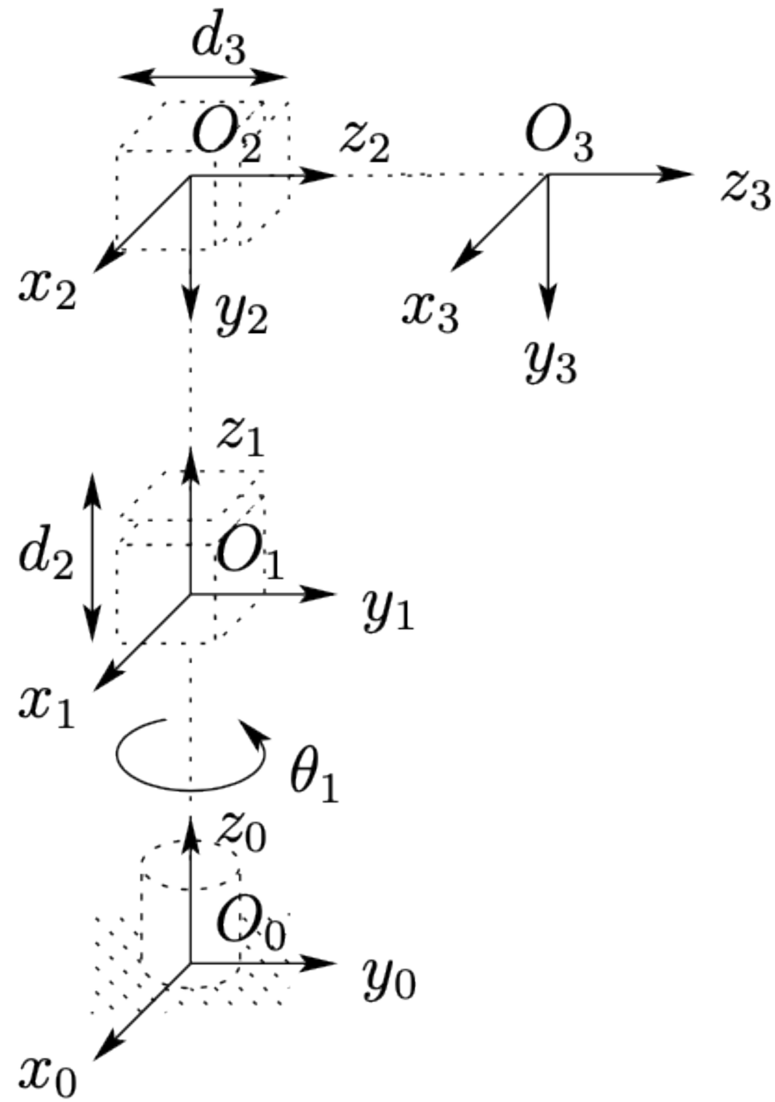
- Joint 0:
  - z_0 is along axis of motion
  - Origin 0 (O_0) is arbitrary, but makes sense
  - x_0 is normal to the page.



Exercise 3:

a. What are the frames?

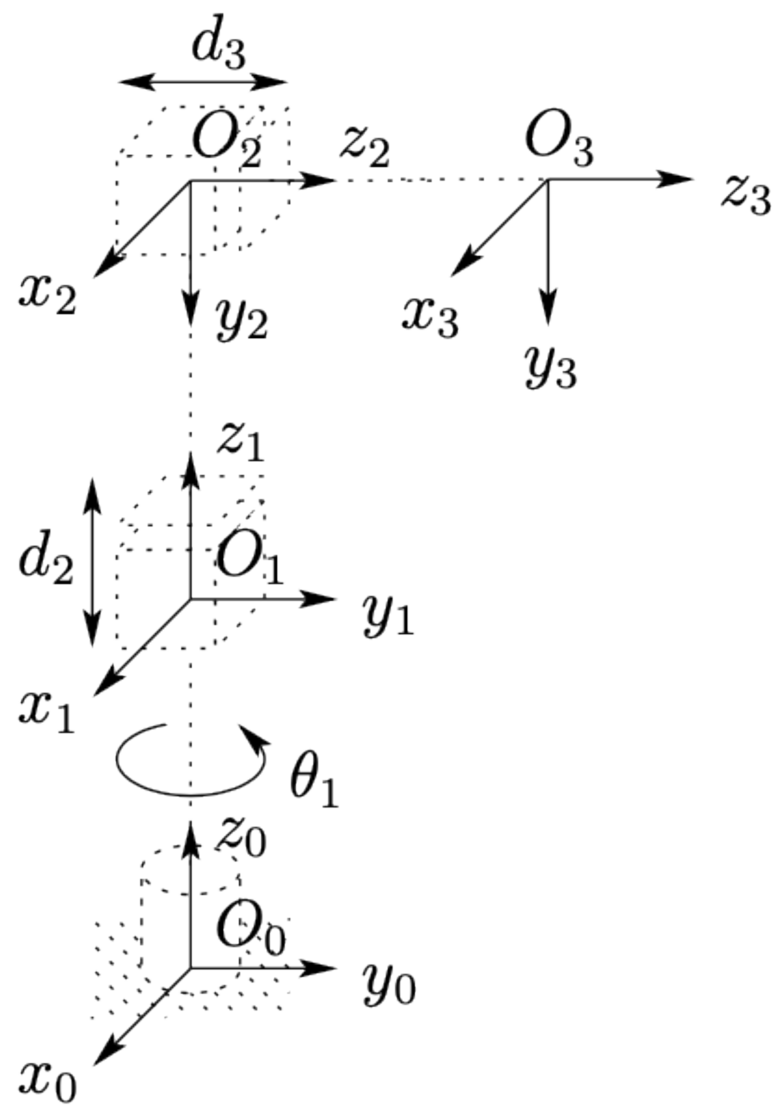
- Joint 1:
  - z_1 is along axis of motion
  - Origin 0 (O_1) is easy, because z_0 and z_1 are the same (no origin movement necessary)
  - x_1 Is normal to the page currently (but not when joint 0 moves!)



Exercise 3:

a. What are the frames?

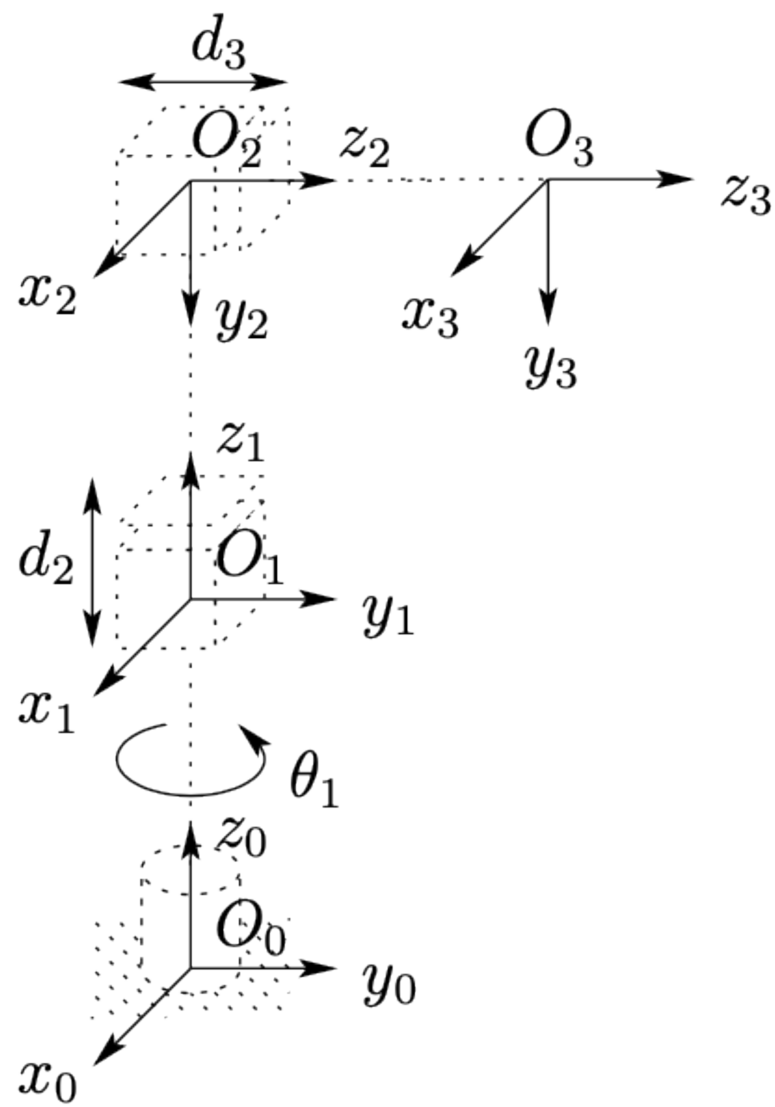
- Joint 2:
  - z_2 is along axis of motion
  - x_2 is chosen parallel to x_1 so that θ_2 is zero.
- Joint 3:
  - Chosen as shown.



Exercise 3:

b. What are the DH parameters?

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1
2	0	-90	d_2	0
3	0	0	d_3	0



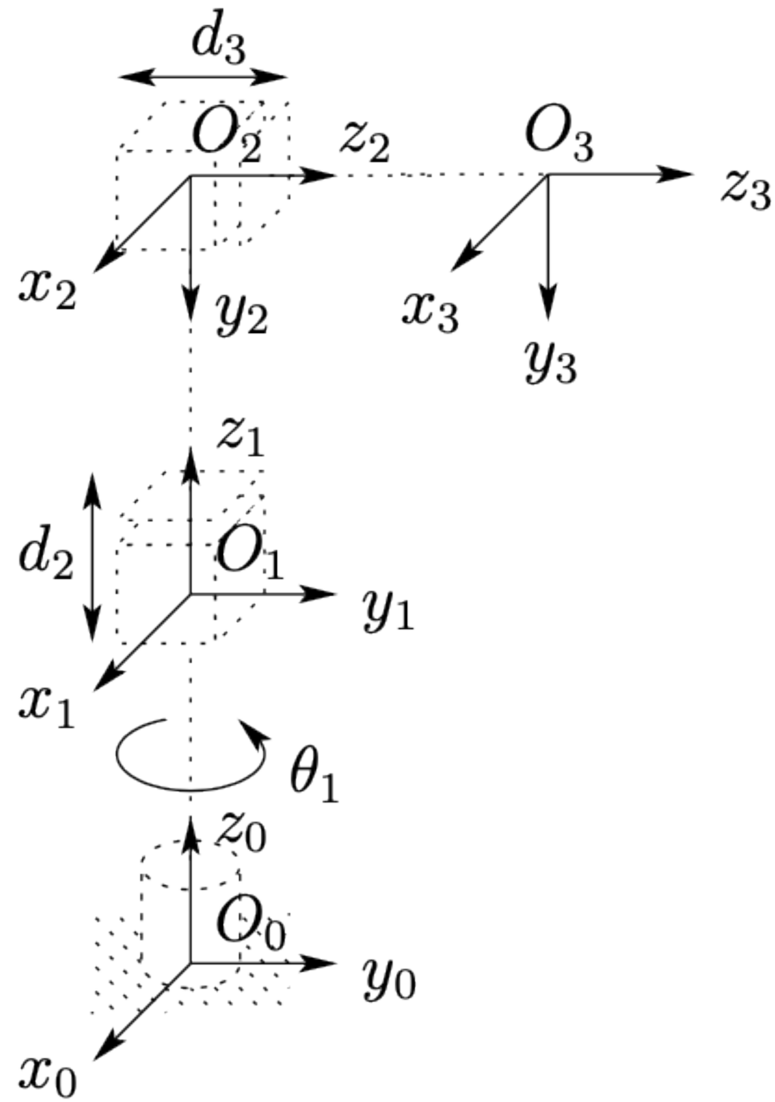
Exercise 3:

c. What are the individual transformation matrices?
   - c = cosine, s = sin

$$^{0}T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

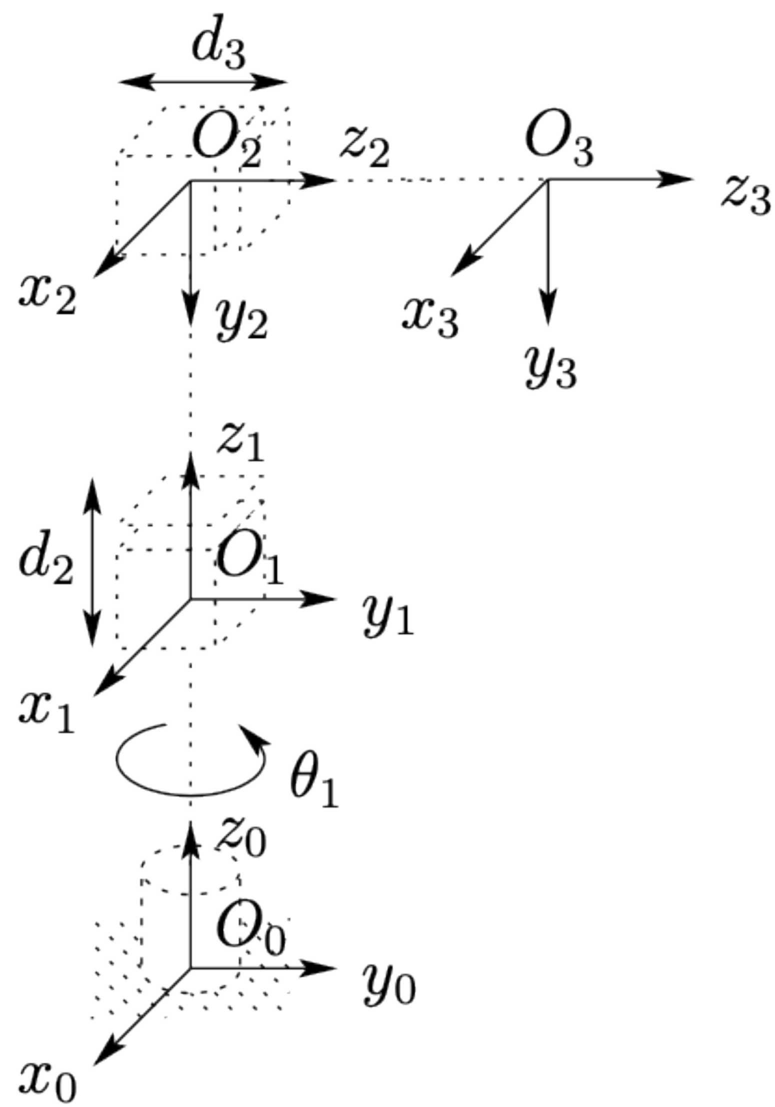
$$^{2}T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Exercise 3:

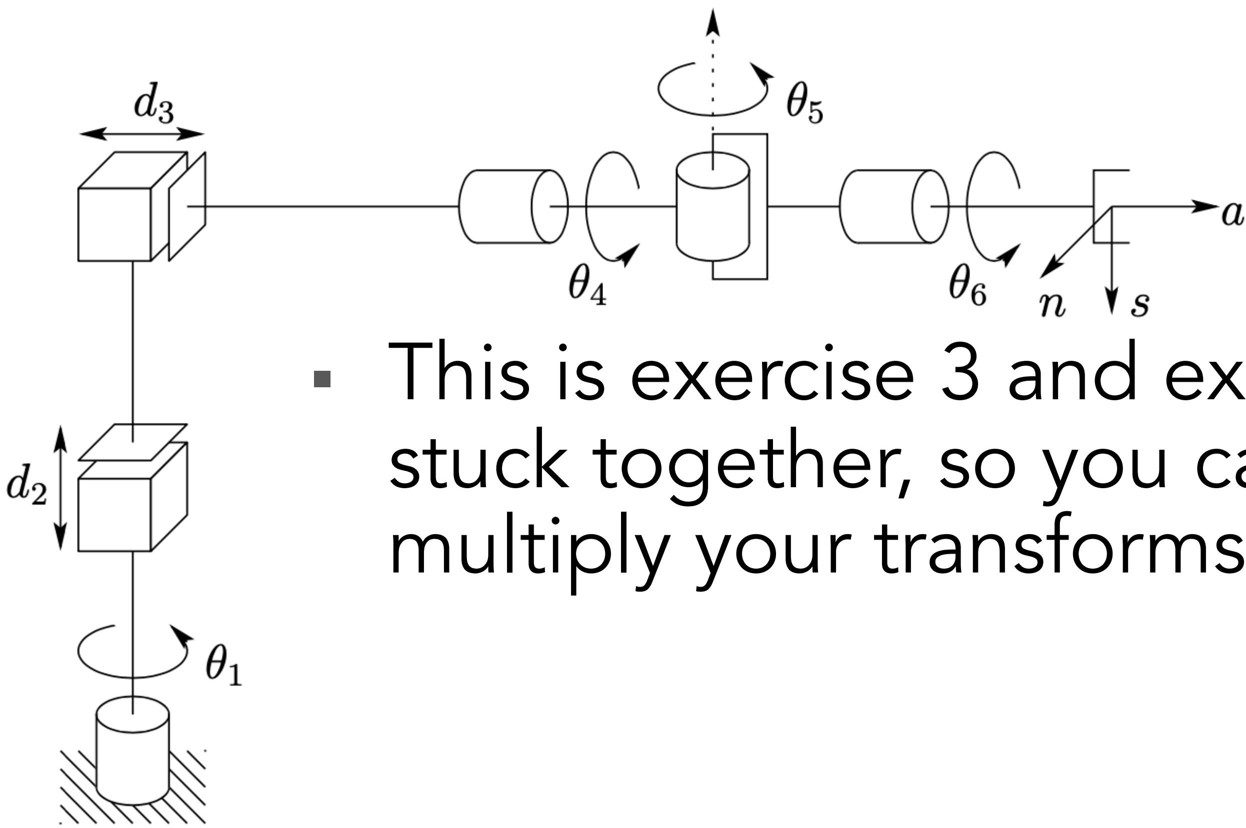
d. What's the final transformation matrix?

$$
{}^{0}T_{3} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Exercise 4:

- What is the complete, derived transformation matrix for this arm? (any approach)



- This is exercise 3 and exercise 2 stuck together, so you can just multiply your transforms for those!