Today

1. Finish kinematics!
   Lots of examples, forward + inverse kinematics.

2. Closed loop control ✓
   - just because I tell robot to move straight doesn't mean it will

3. Thursday - signs of life!!!

4. Homework 1 back
   - view videos

Kinematics

Constraints

$\downarrow$

Rolling
Sliding

$\downarrow$

$$\begin{bmatrix} \text{projection along wheel plane} \\ \text{'' orthog. to '' ''} \end{bmatrix} R(\theta)\,\dot{\xi}_I = \begin{bmatrix} r\dot{\phi} \\ 0 \end{bmatrix}$$

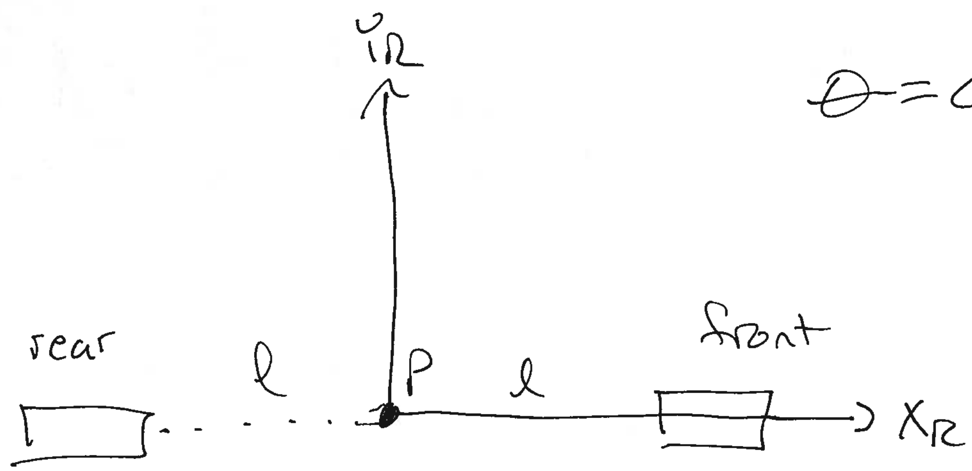
$\downarrow$

System of linear equations

Solve for $\dot{x}, \dot{y}, \dot{\theta}$ = forward kinematics

Solve for $\dot{\phi}$ = inverse kinematics

Bicycle, no steer



$\theta = 0$ for simplicity

rear: $\alpha = \pi$, $\beta = -\frac{\pi}{2}$

front: $\alpha = 0$, $\beta = \frac{\pi}{2}$

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rolling constraints	$\sin(\alpha+\beta)$	$-\cos(\alpha+\beta)$	$-l\cos\beta$	
front :	1	0	0	same
rear :	1	0	0	

Sliding constraints	$\cos(\alpha+\beta)$	$\sin(\alpha+\beta)$	$l\sin\beta$
front :	0	1	l
rear :	0	1	$-l$

Equation 3.28

r_f = radius front
ϕ_f = ang. vel. front

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & l \\ 0 & 1 & -l \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\xi}_I = \begin{bmatrix} r_f \phi_f \\ r_r \phi_r \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & l \\ 0 & 1 & -l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{x} \\ \dot{y} + l\dot{\theta} \\ \dot{y} - l\dot{\theta} \end{bmatrix}$$

$$\left.\begin{aligned} \dot{x} &= r_f \phi_f \\ \dot{x} &= r_r \phi_r \end{aligned}\right] \quad r_f \phi_f = r_r \phi_r$$

$$\left.\begin{aligned} \dot{y} + l\dot{\theta} &= 0 \\ \dot{y} - l\dot{\theta} &= 0 \end{aligned}\right] \Rightarrow \dot{y} = \dot{\theta} = 0$$

Bicycle, steerable front

Rolling constraints

front :	$\sin\beta$	$-\cos\beta$	$-l\cos\beta$
rear :	1	0	0

Sliding constraints

front :	$\cos\beta$	$\sin\beta$	$l\sin\beta$
rear :	0	1	$-l$

$R(\theta)\,\dot{\xi}_I$ for $\theta = 0$! Only holds if aligned with global frame

$$\begin{bmatrix} \sin\beta & -\cos\beta & -l\cos\beta \\ 1 & 0 & 0 \\ \cos\beta & \sin\beta & l\sin\beta \\ 0 & 1 & -l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} =$$

① $\dot{x}\sin\beta - \dot{y}\cos\beta - \dot{\theta}\,l\cos\beta = r_f\dot{\phi}_f$

② $\dot{x} = r_r\dot{\phi}_r \Rightarrow \dot{x} = r_r\dot{\phi}_r$

③ $\dot{x}\cos\beta + \dot{y}\sin\beta + \dot{\theta}\,l\sin\beta = 0$

④ $\dot{y} - l\dot{\theta} = 0 \Rightarrow \dot{y} = l\dot{\theta}$

How ~~What~~ are $r_r \dot{\phi}_r$ and $r_f \dot{\phi}_f$ related?

Multiply ① by $\sin\beta$ and ③ by $\cos\beta$ and add

$$\dot{x}\sin^2\beta + \dot{x}\cos^2\beta = r_f\dot{\phi}_f\sin\beta$$

Use $\dot{x} = r_r\dot{\phi}_r$

$$r_r\dot{\phi}_r(\sin^2\beta + \cos^2\beta) = r_f\dot{\phi}_f\sin\beta$$

$$r_r\dot{\phi}_r = r_f\dot{\phi}_f\sin\beta = \dot{x}$$

If front wheel drive, $\dot{\phi}_r$ depends on $\dot{\phi}_f$

$$\dot{\phi}_r = \frac{r_f}{r_r}\dot{\phi}_f\sin\beta$$

What happens when $\beta \to 0°$? $\dot{\phi}_r \to 0$
Does that make sense?

If rear wheel drive, $\dot{\phi}_f$ depends on $\dot{\phi}_r$

$$\dot{\phi}_f = \frac{r_r\dot{\phi}_r}{r_f\sin\beta}$$

What happens as β approaches $0°$?
Front wheel angular vel approach ∞!! Why?

Assume front wheel drive

Solve ① for $\dot{y}$

$$\dot{x}\sin\beta - \dot{y}\cos\beta - \dot{\theta}l\cos\beta = r_f\dot{\phi}_f$$

$$r_f\dot{\phi}_f\sin^2\beta - \dot{y}\cos\beta - \dot{y}\cos\beta = r_f\dot{\phi}_f$$

$$r_f\dot{\phi}_f(\sin^2\beta - 1) = \dot{y}\,2\cos\beta$$

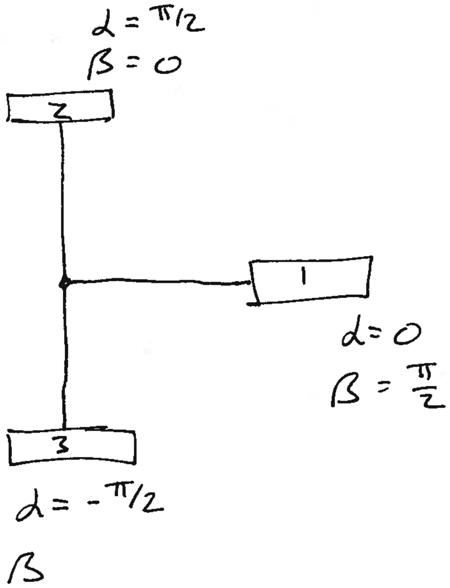
$$\frac{r_f\dot{\phi}_f\cos^2\beta}{2\cos\beta} = \dot{y} \Rightarrow \dot{y} = \frac{1}{2}r_f\dot{\phi}_f\cos\beta$$

Does that make sense? P is ½ way between wheels so half vel.

Skip 2 pages ahead for end

- Evil trike

Tricycle



wheel 3 is ~~drive~~ steer wheel.

Sliding constraints	$\cos(\alpha+\beta)$	$\sin(\alpha+\beta)$	$l \sin\beta$
1:	0	1	l
2:	0	1	0
3:	$\cos(\beta - \frac{\pi}{2}) = \sin\beta$	$\sin(\beta - \frac{\pi}{2}) = -\cos\beta$	$l \sin\beta$

$2 \Rightarrow \dot{y} = 0$

$1 \Rightarrow \dot{y} + l\dot{\theta} = 0 \Rightarrow l\dot{\theta} = 0 \Rightarrow \dot{\theta} = 0$

$3 = \dot{x}\sin\beta - \dot{y}\cos\beta + l\sin\beta\,\dot{\theta} = 0$

$\Rightarrow \dot{x}\sin\beta = 0$

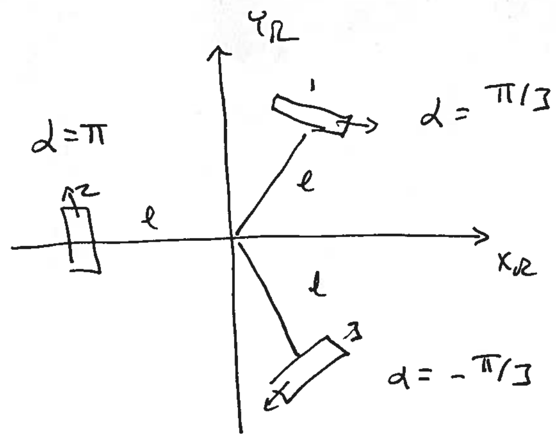
$\Rightarrow \dot{x} = 0 \quad \text{or} \quad \beta = 0 \text{ or } \pi$

$\dot{y} = l\,\dot{\theta}$ so $\dot{\theta} = \frac{\dot{y}}{l} = \frac{1}{2l} r_f \dot{\Omega}_f \cos\beta$

↑ spins more slowly as l increases
Does that make sense?

this goes with pg 4

Omni drive:



$\beta = 0$ for all. which way is positive $\dot\varphi$?

Swedish 90 so $\gamma = 0$!

Constraint along roller axis

$$[\sin(\alpha+\beta+\gamma) \quad -\cos(\alpha+\beta+\gamma) \quad -l\cos(\beta+\gamma)]\, R(\theta)\, \dot\xi_I = r\dot\varphi\cos\gamma$$

$$[\sin\alpha \quad -\cos\alpha \quad -l]\, R(\theta)\, \xi_I = r\dot\varphi$$

1: $\sin\frac{\pi}{3} \quad -\cos\frac{\pi}{3} \quad -l \quad = \frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad -l$

2: $0 \quad 1 \quad -l \quad = 0 \quad 1 \quad -l$

3: $\sin(-\frac{\pi}{3}) \quad -\cos(-\frac{\pi}{3}) \quad -l \quad = -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad -l$

3: $-\sin\frac{2\pi}{3} \quad \cos\frac{2\pi}{3} \quad -l$

Constraints orthogonal to roller axis : NONE!

due to roller spin

if $\theta = 0$ then

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \\ 0 & 1 & -l \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} r_1 \dot{\phi}_1 \\ r_2 \dot{\phi}_2 \\ r_3 \dot{\phi}_3 \end{bmatrix}$$

assume $r_1 = r_2 = r_3 = 1$

$$\frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} - l\dot{\theta} = \dot{\phi}_1$$

$$\dot{y} - l\dot{\theta} = \dot{\phi}_2$$

$$-\frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} - l\dot{\theta} = \dot{\phi}_3$$

Inverse kinematics!!

Suppose I want to translate along X at a rate of 2?

$\dot{x} = 2, \quad \dot{y} = 0, \quad \dot{\theta} = 0$

Using equations above I get

$$\frac{\sqrt{3}}{2} \cdot 2 - \frac{1}{2}0 - l0 = \dot{\phi}_1 = \sqrt{3}$$

$$0 - l0 = \dot{\phi}_2 = 0$$

$$-\frac{\sqrt{3}}{2} 2 - \frac{1}{2}0 - l0 = \dot{\phi}_3 = -\sqrt{3}$$

Does this make sense?

Suppose I want $\dot{y} = 2$ & $\dot{x} = 0$, $\dot{\theta} = 0$?

$$\left.\begin{aligned} \dot{d}_1 &= -1 \\ \dot{d}_2 &= 2 \\ \dot{d}_3 &= -1 \end{aligned}\right]$$

Does this make sense?

What about pure rotation?

$$\dot{d}_1 = \dot{d}_2 = \dot{d}_3 = -l\dot{\theta}$$

No matter what values of $\dot{x}, \dot{y}, \dot{\theta}$, I can compute $\dot{d}_1, \dot{d}_2, \dot{d}_3$

Forward Kinematics
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(1) $\frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} - l\dot{\theta} = \dot{d}_1$

(2) $\dot{y} - l\dot{\theta} = \dot{d}_2$

(3) $-\frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} - l\dot{\theta} = \dot{d}_3$

Add (1) + (3)

(4) $-\dot{y} - 2l\dot{\theta} = \dot{d}_1 + \dot{d}_3$

Add (2) + (4)

$$-3l\dot{\theta} = \dot{d}_1 + \dot{d}_2 + \dot{d}_3$$

$$\Rightarrow \boxed{\dot{\theta} = -\frac{\dot{d}_1 + \dot{d}_2 + \dot{d}_3}{3l}}$$

Substitute into 2 to get

$$\dot{y} - l\dot{\theta} = \dot{d}_2$$

$$\dot{y} = \dot{d}_2 + l\dot{\theta}$$

$$\dot{y} = \dot{d}_2 + l\left(-\frac{\dot{d}_1 + \dot{d}_2 + \dot{d}_3}{3l}\right)$$

$$= \dot{d}_2 - \frac{1}{3}\dot{d}_1 - \frac{1}{3}\dot{d}_2 - \frac{1}{3}\dot{d}_3$$

$$\boxed{\dot{y} = \frac{2}{3}\dot{d}_2 - \frac{1}{3}\dot{d}_1 - \frac{1}{3}\dot{d}_3}$$

Substitute into 1 to get

$$\frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} - l\dot{\theta} = \dot{d}_1$$

$$\dot{x} = \left(\dot{d}_1 + \frac{1}{2}\dot{y} + l\dot{\theta}\right)\frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left(\dot{d}_1 + \frac{2}{6}\dot{d}_2 - \frac{1}{6}\dot{d}_1 - \frac{1}{6}\dot{d}_3 - \frac{1}{3}\dot{d}_1 - \frac{1}{3}\dot{d}_2 - \frac{1}{3}\dot{d}_3\right)$$

$$\boxed{\dot{x} = \frac{2}{\sqrt{3}}\left(\frac{1}{2}\dot{d}_1 - \frac{1}{2}\dot{d}_3\right)}$$