First-Order Logic: Review

First-order logic

- First-order logic (FOL) models the world in terms of
  - **Objects,** which are things with individual identities
  - **Properties** of objects that distinguish them from others
  - **Relations** that hold among sets of objects
  - **Functions,** which are a subset of relations where there is only one "value" for any given "input"
- Examples:
  - Objects: Students, lectures, companies, cars ...
  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, more-than ...

User provides

- **Constant symbols** representing individuals in the world
  - Mary, 3, green
- **Function symbols,** map individuals to individuals
  - father_of(Mary) = John
  - color_of(Sky) = Blue
- **Predicate symbols,** map individuals to truth values
  - greater(5,3)
  - green(Grass)
  - color(Grass, Green)

FOL Provides

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in propositional logic: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), iff ($\leftrightarrow$)
- **Quantifiers**
  - Universal **$\forall$x** or **(Ax)**
  - Existential **$\exists$x** or **(Ex)**

Sentences: built from terms and atoms

- A **term** (denoting a real-world individual) is a constant symbol, variable symbol, or n-place function of n terms, e.g.:
  - Constants: john, umbc
  - Variables: x, y, z
  - Functions: mother_of(john), phone(mother(x))
- Ground terms have no variables in them
  - **Ground:** john, father_of(father_of(john))
  - **Not Ground:** father_of(X)

Sentences: built from terms and atoms

- An **atomic sentence** (which has value true or false) is an n-place predicate of n terms, e.g.:
  - green(Kermit))
  - between(Philadelphia, Baltimore, DC)
  - loves(X, mother(X))
- A **complex sentence** is formed from atomic sentences connected by logical connectives:

  $$\neg P,\ P \vee Q,\ P \wedge Q,\ P \rightarrow Q,\ P \leftrightarrow Q$$

  where P and Q are sentences

Sentences: built from terms and atoms

- **quantified sentences** adds quantifiers ∀ and ∃
  - ∀x loves(x, mother(x))
  - ∃x number(x) ∧ greater(x, 100), prime(x)
- A **well-formed formula (wff)** is a sentence containing no "free" variables, i.e., all variables are "bound" by either a universal or existential quantifiers

  (∀x)P(x,y) has x bound as a universally quantified variable, but y is free

A BNF for FOL

```
S := <Sentence> ;
<Sentence> := <AtomicSentence> |
          <Sentence> <Connective> <Sentence> |
          <Quantifier> <Variable>,... <Sentence> |
          "NOT" <Sentence> |
          "(" <Sentence> ")";
<AtomicSentence> := <Predicate> "(" <Term>, ... ")" |
                    <Term> "=" <Term>;
<Term> := <Function> "(" <Term>, ... ")" |
          <Constant> |
          <Variable>;
<Connective> := "AND" | "OR" | "IMPLIES" | "EQUIVALENT";
<Quantifier> := "EXISTS" | "FORALL" ;
<Constant> := "A" | "X1" | "John" | ... ;
<Variable> := "a" | "x" | "s" | ... ;
<Predicate> := "Before" | "HasColor" | "Raining" | ... ;
<Function> := "Mother" | "LeftLegOf" | ... ;
```

Quantifiers

- **Universal quantification**
  - $(\forall x)P(x)$ means P holds for **all** values of x in domain associated with variable
  - E.g., $(\forall x)$ dolphin(x) $\rightarrow$ mammal(x)
- **Existential quantification**
  - $(\exists\, x)P(x)$ means P holds for **some** value of x in domain associated with variable
  - E.g., $(\exists\, x)$ mammal(x) $\wedge$ lays_eggs(x)
  - This lets us make a statement about some object without naming it

Quantifiers

- Universal quantifiers often used with *implies* to form *rules*:

  $(\forall x)\ \text{student}(x) \rightarrow \text{smart}(x)$ means "All students are smart"

- Universal quantification *rarely* used to make blanket statements about every individual in the world:

  $(\forall x)\text{student}(x) \wedge \text{smart}(x)$ means "Everyone in the world is a student and is smart"

- Existential quantifiers usually used with "and" to specify a list of properties about an individual:

  $(\exists x)\ \text{student}(x) \wedge \text{smart}(x)$ means "There is a student who is smart"

- Common mistake: represent this sentence in FOL as:

  $(\exists x)\ \text{student}(x) \rightarrow \text{smart}(x)$

  – What does this sentence mean?

Quantifier Scope

- FOL sentences have structure, like programs
- In particular, the variables in a sentence have a scope
- For example, suppose we want to say
  - "everyone who is alive loves someone"
  - $(\forall x)\ alive(x) \rightarrow (\exists y)\ loves(x,y)$
- Here's how we scope the variables

$$(\forall x)\ alive(x) \rightarrow (\exists y)\ loves(x,y)$$

Scope of x

Scope of y

Quantifier Scope

- **Switching order of universal quantifiers *does not* change the meaning**
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x) P(x,y)$
  - "Dogs hate cats" (i.e., "all dogs hate all cats")
- **You can switch order of existential quantifiers**
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x) P(x,y)$
  - "A cat killed a dog"
- **Switching order of universals and existentials *does* change meaning:**
  - Everyone likes someone: $(\forall x)(\exists y)$ likes(x,y)
  - Someone is liked by everyone: $(\exists y)(\forall x)$ likes(x,y)

Connections between All and Exists

- We can relate sentences involving $\forall$ and $\exists$ using extensions to **De Morgan's laws**:
  1. $(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$
  2. $\neg(\forall x)\ P \leftrightarrow (\exists x)\ \neg P(x)$
  3. $(\forall x)\ P(x) \leftrightarrow \neg\ (\exists x)\ \neg P(x)$
  4. $(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$
- Examples
  1. All dogs don't like cats $\leftrightarrow$ No dogs like cats
  2. Not all dogs dance $\leftrightarrow$ There is a dog that doesn't dance
  3. All dogs sleep $\leftrightarrow$ There is no dog that doesn't sleep
  4. There is a dog that talks $\leftrightarrow$ Not all dogs can't talk

Quantified inference rules

- Universal instantiation
  - $\forall x\ P(x)\ \therefore\ P(A)$ # *where A is some constant*
- Universal generalization
  - $P(A) \wedge P(B) \ldots \therefore\ \forall x\ P(x)$ # *if AB… enumerate all*
    # *individuals*
- Existential instantiation
  - $\exists x\ P(x)\ \therefore P(F)$

←**Skolem* constant F**
F must be a "new" constant not appearing in the KB

- Existential generalization
  - $P(A)\ \therefore\ \exists x\ P(x)$

* After [Thoralf Skolem]

Universal instantiation (a.k.a. universal elimination)

- If $(\forall x)\ P(x)$ is true, then $P(C)$ is true, where C is *any* constant in the domain of x, e.g.:

  $(\forall x)$ eats(John, x) $\Rightarrow$
  eats(John, Cheese18)

- Note that function applied to ground terms is also a constant

  $(\forall x)$ eats(John, x) $\Rightarrow$
  eats(John, contents(Box42))

Existential instantiation (a.k.a. existential elimination)

- From $(\exists x)\ P(x)$ infer $P(c)$, e.g.:
  - $(\exists x)$ eats(Mikey, x) $\rightarrow$ eats(Mikey, Stuff345)
- The variable is replaced by a **brand-new constant** not occurring in this or any sentence in the KB
- Also known as skolemization; constant is a **skolem constant**
- We don't want to accidentally draw other inferences about it by introducing the constant
- Can use this to reason about unknown objects, rather than constantly manipulating existential quantifiers

Existential generalization (a.k.a. existential introduction)

- If $P(c)$ is true, then $(\exists x)\ P(x)$ is inferred, e.g.:

  $$\text{Eats(Mickey, Cheese18)} \Rightarrow$$
  $$(\exists x)\ \text{eats(Mickey, x)}$$

- All instances of the given constant symbol are replaced by the new variable symbol
- Note that the variable symbol cannot already exist anywhere in the expression

Translating English to FOL

Every gardener likes the sun

$\forall x\ gardener(x) \rightarrow likes(x,Sun)$

You can fool some of the people all of the time

$\exists x\ \forall t\ \ person(x) \land time(t) \rightarrow can\text{-}fool(x, t)$

You can fool all of the people some of the time

$\exists t\ time(t) \land \forall x\ person(x) \rightarrow can\text{-}fool(x, t)$

$\forall x\ person(x) \rightarrow \exists t\ time(t) \land can\text{-}fool(x, t)$

Note 2 possible readings of NL sentence

All purple mushrooms are poisonous

$\forall x\ (mushroom(x) \land purple(x)) \rightarrow poisonous(x)$

Translating English to FOL

No purple mushroom is poisonous (two ways)

$\neg\exists x\ purple(x) \wedge mushroom(x) \wedge poisonous(x)$

$\forall x\ \ (mushroom(x) \wedge purple(x)) \rightarrow \neg poisonous(x)$

There are exactly two purple mushrooms

$\exists x\ \exists y\ mushroom(x) \wedge purple(x) \wedge mushroom(y) \wedge purple(y) \wedge \neg(x=y) \wedge \forall z\ (mushroom(z) \wedge purple(z)) \rightarrow ((x=z) \vee (y=z))$

Obama is not short

$\neg short(Obama)$

Logic and People

"Logic—the last refuge of a scoundrel."

- People can easily be confused by logic
- And are often suspicious of it, or give it too much weight

Monty Python example (Russell & Norvig)

FIRST VILLAGER: We have found a witch. May we burn her?
ALL: A witch! Burn her!
BEDEVERE: Why do you think she is a witch?
SECOND VILLAGER: She turned *me* into a newt.
B: A newt?
V2 *(after looking at himself for some time)*: I got better.
ALL: Burn her anyway.
B: Quiet! Quiet! There are ways of telling whether she is a witch.

B: Tell me… what do you do with witches?

ALL: Burn them!

B: And what do you burn, apart from witches?

V4: …wood?

B: So **why do witches burn**?

V2 *(pianissimo)*: **because they' re made of wood?**

B: Good.

ALL: I see. Yes, of course.

B: So how can we tell if she is made of wood?

V1: Make a bridge out of her.

B: Ah… but can you not also make bridges out of stone?

ALL: Yes, of course… um… er…

B: Does wood sink in water?

ALL: No, no, it floats. Throw her in the pond.

B: Wait. Wait… tell me, what also floats on water?

ALL: Bread? No, no no. Apples… gravy… very small rocks…

B: No, no, no,

KING ARTHUR: A duck!

(They all turn and look at Arthur. Bedevere looks up, very impressed.)

B: Exactly. So… logically…

V1 *(beginning to pick up the thread)*: **If she… weighs the same as a duck… she's made of wood**.

B: And therefore?

ALL: A witch!

Fallacy: Affirming the conclusion

$\forall x \; witch(x) \rightarrow burns(x)$

$\forall x \; wood(x) \rightarrow burns(x)$

$\therefore \; \forall z \; witch(x) \rightarrow wood(x)$

$p \rightarrow q$

$r \rightarrow q$

$p \rightarrow r$

Monty Python Near-Fallacy #2

$\text{wood}(x) \rightarrow \text{can-build-bridge}(x)$

$\therefore\ \text{can-build-bridge}(x) \rightarrow \text{wood}(x)$

- B: Ah… but can you not also make bridges out of stone?

Monty Python Fallacy #3

$\forall x$ wood(x) $\rightarrow$ floats(x)

$\forall x$ duck-weight (x) $\rightarrow$ floats(x)

$\therefore$ $\forall x$ duck-weight(x) $\rightarrow$ wood(x)

$p \rightarrow q$

$r \rightarrow q$

$\therefore$ $r \rightarrow p$

Monty Python Fallacy #4

$\forall z\ light(z) \rightarrow wood(z)$

light(W)

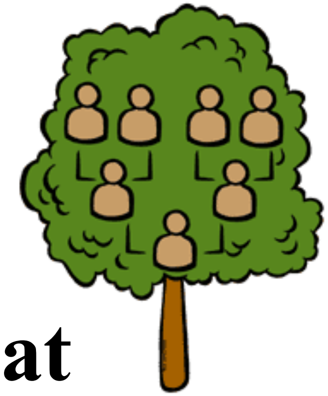
$\therefore$ wood(W) % ok…………..

witch(W) $\rightarrow$ wood(W) % applying universal instan.
% to fallacious conclusion #1

wood(W)

$\therefore$ witch(z)

Simple genealogy KB in FOL



Design a knowledge base using FOL that

- Has facts of immediate family relations, e.g., spouses, parents, etc.
- Has definitions of more complex relations (ancestors, relatives)
- Can detect conflicts, e.g., you are your own parent
- Can infer relations, e.g., grandparernt from parent
- Can answer queries about relationships between people

How do we approach this?

- Design an initial ontology of types, e.g.
  - e.g., person, man, woman, gender
- Add general individuals to ontology, e.g.
  - gender(male), gender(female)
- Extend ontology be defining relations, e.g.
  - spouse, has_child, has_parent
- Add general constraints to relations, e.g.
  - spouse(X,Y) => ~ X = Y
  - spouse(X,Y) => person(X), person(Y)
- Add FOL sentences for inference, e.g.
  - spouse(X,Y) ⇔ spouse(Y,X)
  - man(X) ⇔ person(X) ∧ gender(X, male)

Simple genealogy KB in FOL

People knowledge base using FOL that

- Has facts of immediate family relations, e.g., spouses, parents, etc.
- Has definitions of more complex relations (ancestors, relatives)
- Can detect conflicts, e.g., you are your own parent
- Can infer relations, e.g., grandparernt from parent
- Can answer queries about relationships between people

Example: A simple genealogy KB by FOL

- **Build a small genealogy knowledge base using FOL that**
  - contains facts of immediate family relations (spouses, parents, etc.)
  - contains definitions of more complex relations (ancestors, relatives)
  - is able to answer queries about relationships between people
- **Predicates:**
  - parent(x, y), child(x, y), father(x, y), daughter(x, y), etc.
  - spouse(x, y), husband(x, y), wife(x,y)
  - ancestor(x, y), descendant(x, y)
  - male(x), female(y)
  - relative(x, y)
- **Facts:**
  - husband(Joe, Mary), son(Fred, Joe)
  - spouse(John, Nancy), male(John), son(Mark, Nancy)
  - father(Jack, Nancy), daughter(Linda, Jack)
  - daughter(Liz, Linda)
  - etc.

- **Rules for genealogical relations**

  $(\forall x,y)$ parent(x, y) $\leftrightarrow$ child (y, x)
  $(\forall x,y)$ father(x, y) $\leftrightarrow$ parent(x, y) $\wedge$ male(x) *;similarly for mother(x, y)*
  $(\forall x,y)$ daughter(x, y) $\leftrightarrow$ child(x, y) $\wedge$ female(x) *;similarly for son(x, y)*
  $(\forall x,y)$ husband(x, y) $\leftrightarrow$ spouse(x, y) $\wedge$ male(x) *;similarly for wife(x, y)*
  $(\forall x,y)$ spouse(x, y) $\leftrightarrow$ spouse(y, x) *;spouse relation is symmetric*
  $(\forall x,y)$ parent(x, y) $\rightarrow$ ancestor(x, y)
  $(\forall x,y)(\exists z)$ parent(x, z) $\wedge$ ancestor(z, y) $\rightarrow$ ancestor(x, y)
  $(\forall x,y)$ descendant(x, y) $\leftrightarrow$ ancestor(y, x)
  $(\forall x,y)(\exists z)$ ancestor(z, x) $\wedge$ ancestor(z, y) $\rightarrow$ relative(x, y)
  ;related by common ancestry
  $(\forall x,y)$ spouse(x, y) $\rightarrow$ relative(x, y) ;related by marriage
  $(\forall x,y)(\exists z)$ relative(z, x) $\wedge$ relative(z, y) $\rightarrow$ relative(x, y) *;transitive*
  $(\forall x,y)$ relative(x, y) $\leftrightarrow$ relative(y, x) *;symmetric*

- **Queries**
  - ancestor(Jack, Fred) *; the answer is yes*
  - relative(Liz, Joe) *; the answer is yes*
  - relative(Nancy, Matthew) *;no answer, no under closed world assumption*
  - $(\exists z)$ ancestor(z, Fred) $\wedge$ ancestor(z, Liz)

Axioms for Set Theory in FOL

1. The only sets are the empty set and those made by adjoining something to a set:

   $\forall s\ \text{set}(s) \Leftrightarrow (s=\text{EmptySet}) \lor (\exists x,r\ \text{Set}(r) \land s=\text{Adjoin}(s,r))$

2. The empty set has no elements adjoined to it:

   $\sim \exists x,s\ \text{Adjoin}(x,s)=\text{EmptySet}$

3. Adjoining an element already in the set has no effect:

   $\forall x,s\ \text{Member}(x,s) \Leftrightarrow s=\text{Adjoin}(x,s)$

4. The only members of a set are the elements that were adjoined into it:

   $\forall x,s\ \text{Member}(x,s) \Leftrightarrow \exists y,r\ (s=\text{Adjoin}(y,r) \land (x=y \lor \text{Member}(x,r)))$

5. A set is a subset of another iff all of the 1st set's members are members of the 2^{nd}:

   $\forall s,r\ \text{Subset}(s,r) \Leftrightarrow (\forall x\ \text{Member}(x,s) \Rightarrow \text{Member}(x,r))$

6. Two sets are equal iff each is a subset of the other:

   $\forall s,r\ (s=r) \Leftrightarrow (\text{subset}(s,r) \land \text{subset}(r,s))$

7. Intersection

   $\forall x,s1,s2\ \text{member}(X,\text{intersection}(S1,S2)) \Leftrightarrow \text{member}(X,s1) \land \text{member}(X,s2)$

8. Union

   $\exists x,s1,s2\ \text{member}(X,\text{union}(s1,s2)) \Leftrightarrow \text{member}(X,s1) \lor \text{member}(X,s2)$

Semantics of FOL

- **Domain M**: the set of all objects in the world (of interest)
- **Interpretation I**: includes
  - Assign each constant to an object in M
  - Define each function of n arguments as a mapping M^n => M
  - Define each predicate of n arguments as a mapping M^n => {T, F}
  - Therefore, every ground predicate with any instantiation will have a truth value
  - In general there's an infinite number of interpretations because |M| is infinite
- **Define logical connectives**: **~, ^, v, =>, <=>** as in PL
- **Define semantics of (∀x) and (∃x)**
  - (∀x) P(x) is true iff P(x) is true under all interpretations
  - (∃x) P(x) is true iff P(x) is true under some interpretation

- **Model**: an interpretation of a set of sentences such that every sentence is *True*
- **A sentence is**
  - **satisfiable** if it is true under some interpretation
  - **valid** if it is true under all possible interpretations
  - **inconsistent** if there does not exist any interpretation under which the sentence is true
- **Logical consequence**: S |= X if all models of S are also models of X

Axioms, definitions and theorems

- **Axioms** are facts and rules that attempt to capture the (important) facts and concepts about a domain; axioms can be used to prove **theorems**
  - Mathematicians dislike unnecessary (dependent) axioms, i.e. ones that can be derived from others
  - Dependent axioms can make reasoning faster, however
  - Choosing a good set of axioms is a design problem
- A **definition** of a predicate is of the form "p(X) $\leftrightarrow$ …" and can be decomposed into two parts
  - **Necessary** description: "p(x) $\rightarrow$ …"
  - **Sufficient** description "p(x) $\leftarrow$ …"
  - Some concepts don't have complete like …

More on definitions

Example: define father(x, y) by parent(x, y) and male(x)

- **parent(x, y)** is a necessary (but not sufficient) description of father(x, y)

  father(x, y) → parent(x, y)

- **parent(x, y) ^ male(x) ^ age(x, 35)** is a sufficient (but not necessary) description of father(x, y):

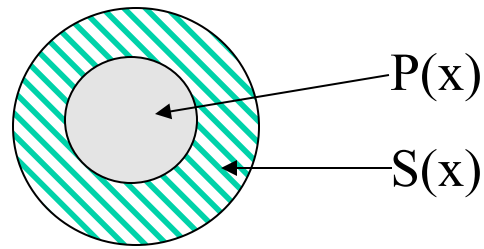
  father(x, y) ← parent(x, y) ^ male(x) ^ age(x, 35)

- **parent(x, y) ^ male(x)** is a necessary and sufficient description of father(x, y)

  parent(x, y) ^ male(x) ↔ father(x, y)

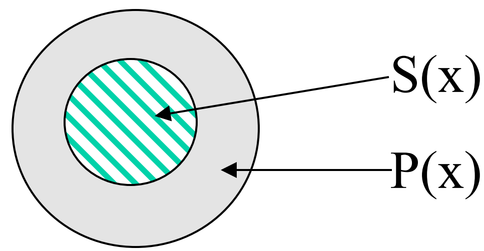
More on definitions

S(x) is a necessary condition of P(x)



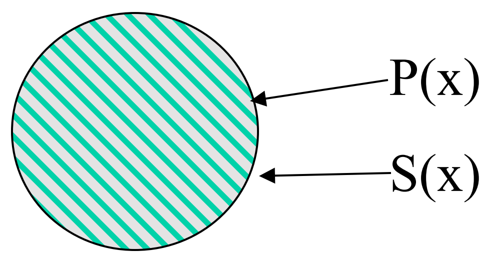
$(\forall x)\ P(x) \Rightarrow S(x)$

S(x) is a sufficient condition of P(x)



$(\forall x)\ P(x) \Leftarrow S(x)$

S(x) is a necessary and sufficient condition of P(x)



$(\forall x)\ P(x) \Leftrightarrow S(x)$

Higher-order logic

- FOL only lets us quantify over variables, and variables can only range over objects.
- HOL allows us to quantify over relations, e.g.

  "two functions are equal iff they produce the same value for all arguments"

  $\forall f \, \forall g \, (f = g) \leftrightarrow (\forall x \; f(x) = g(x))$

- E.g.: (quantify over predicates)

  $\forall r \; \text{transitive}(\, r\,) \rightarrow (\forall xyz)\; r(x,y) \wedge r(y,z) \rightarrow r(x,z))$

- More expressive, but undecidable, in general

Expressing uniqueness

syntactic sugar

- Often want to say that there is a single, unique object that satisfies a condition
- There exists a unique x such that king(x) is true
  - $\exists x\ king(x) \wedge \forall y\ (king(y) \rightarrow x=y)$
  - $\exists x\ king(x) \wedge \neg \exists y\ (king(y) \wedge x \neq y)$
  - $\exists!\ x\ king(x)$
- "Every country has exactly one ruler"
  - $\forall c\ country(c) \rightarrow \exists!\ r\ ruler(c,r)$
- Iota operator: $\iota\ x\ P(x)$ means "the unique x such that p(x) is true"
  - "The unique ruler of Freedonia is dead"
  - $dead(\iota\ x\ ruler(freedonia,x))$

Notational differences

- **Different symbols** for *and, or, not, implies, ...*
  - $\forall \quad \exists \quad \Rightarrow \quad \Leftrightarrow \quad \wedge \quad \vee \quad \neg \quad \bullet \quad \supset$
  - p v (q ^ r)
  - p + (q * r)
- **Prolog**

  cat(X) :- furry(X), meows (X), has(X, claws)
- **Lispy notations**

```
(forall ?x (implies (and (furry ?x)
                         (meows ?x)
                         (has ?x claws))
                    (cat ?x)))
```

A example of FOL in use

- Semantics of W3C's semantic web stack (RDF, RDFS, OWL) is defined in FOL
- OWL Full is equivalent to FOL
- Other OWL profiles support a subset of FOL and are more efficient
- However, the semantics of schema.org is only defined in natural language text
- …And Google's knowledge Graph probably (!) uses probabilities

FOL Summary

- First order logic (FOL) introduces predicates, functions and quantifiers
- More expressive, but reasoning is more complex
  - Reasoning in propositional logic is NP hard, FOL is semi-decidable
- A common AI knowledge representation language
  - Other KR languages (e.g., OWL) are often defined by mapping them to FOL
- FOL variables range over objects
  - HOL variables can range over functions, predicates or sentences