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CHAPTER 1

MATHEMATICAL PRELIMINARIES

" Wavelets, what kind of signal processing is that?" — *Bruce Suter (1996)*

In this chapter we introduce the relevant mathematics necessary for an understanding of wavelets. This will serve the dual purpose of making the reader familiar with the mathematics in common use in the wavelet literature and fourier analysis. We present proofs of many of the theorems in the hope that they will illustrate some of the manipulative techniques necessary for success in this field. Throughout this chapter we will refer to $\mathcal{V}$ as an arbitrary vector space, $\Re$ as the set of real numbers, and $\mathcal{C}$ as the set of complex numbers. When we perform integration, if there are no limits, we assume that they go from $-\infty$ to $+\infty$.

Mathematical Tomfoolery

Definition 1 (Vector Space). *A vector (linear) space $\mathcal{V}$ over a field $\mathcal{F}$ is a set of elements together with a function, called addition, and a function called scalar multiplication which satisfy the following conditions for all $f,\ g,\ h\ \in \mathcal{V}$ and $\alpha,\ \beta\ \in \mathcal{F}$:*

1. $(f+g)\ +\ h = f\ +\ (g+h)$
2. $f+\ g=\ g+\ f$
3. *There exists an element* 0 *in* $\mathcal{V}$ *such that* $f\ +\ 0\ =\ f$ *for all* $f\ \in\ \mathcal{V}$
4. *For each* $f\ \in\ \mathcal{V}$ *there is an element* $-x\ \in\ \mathcal{V}$ *s.t.* $x\ +\ (-x)\ =\ 0$
5. $\alpha(f\ +\ g) = \alpha f\ +\ \alpha g$
6. $(\alpha\ +\beta)f\ = \alpha f\ +\ \beta f$
7. $\alpha(\beta f)\ = (\alpha\beta)f$
8. $1\ f\ = f$ *(rememembering that there exists an element 1 in the field).*

For example we could have a vector space over the field of complex numbers. A complex vector space (as opposed to a vector space over a complex field) is a vector space where we allow complex numbers for scalar multiplication.

Definition 2 (Inner Product). *If $\mathcal{V}$ is a complex vector space, then an inner product is a mapping $\langle f,\ g\rangle : \mathcal{V}\times\mathcal{V}\rightarrow\mathcal{C}$ with the following properties:*

1. $\langle f, ag\ +bh\rangle = a\langle f,g\rangle + b\langle f,h\rangle$
2. $\langle ag\ +bh, f\rangle = a^*\langle g,f\rangle + b^*\langle g,h\rangle$
3. $\langle f,g\rangle = \overline{\langle g,f\rangle}$
4. $\langle f,f\rangle \geq 0$ *and* $\langle f,f\rangle = 0$ *implies* $f=0$.

Theorem 1 (Inner product spaces). *The space $\mathcal{V}$ of continuous functions $f:\Re\rightarrow\mathcal{C}$ forms an inner product space if we define:*

$$\langle f,g\rangle = \int_{-\infty}^{+\infty} f(x)^* g(x)\ dx \qquad (01\text{–}001\ )$$

Proof: Easy. Just substitute the definition of inner product into the various definitions. ♣

Now we present two important properties of functions that belong to an inner product space $\mathcal{V}$. We will use these in the proof of the uncertainty principle.

Theorem 2 (Cauchy–Bunyakovski-Schwarz (CBS) inequality). *If $\mathcal{V}$ is an inner product space, then for all $f, g \in \mathcal{V}$*

$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle\langle g, g\rangle \qquad (01\text{–}002\ )$$

Proof: Start with $0 \leq \langle f - \lambda g, f - \lambda g\rangle$ where λ is complex and is defined by $\lambda = \langle f, g\rangle / \langle g, g\rangle$. Now expand the inner product as:

$$0 \leq \int_{-\infty}^{+\infty} (f - \lambda g)^*(f - \lambda g)\ dt$$
$$0 \leq \langle f, f\rangle - \lambda^*\langle g, f\rangle - \lambda\langle f, g\rangle + |\lambda|^2\langle g, g\rangle.$$

Substituting for λ we get

$$0 \leq \langle f, f\rangle - \frac{|\langle f, f\rangle^*|^2}{\langle g, g\rangle}.$$

For a complex number A, $|A^*|^2 = |A|^2$, leading to the CBS inequality. ♣

Theorem 3 (Minkowski inequality). *If $\mathcal{V}$ is an inner product space, then for all $f, g \in \mathcal{V}$*

$$\langle f + g, f + g\rangle^{\frac{1}{2}} \leq \langle f, f\rangle^{\frac{1}{2}} + \langle g, g\rangle^{\frac{1}{2}}.$$

Proof:

$$\begin{aligned}\langle f + g, f + g\rangle &= \int_{-\infty}^{+\infty} (f + g)^*(f + g)\ dt \\ &= \langle f, f\rangle + \langle f, g\rangle + \langle f, g\rangle^* + \langle g, g\rangle \\ &= \langle f, f\rangle + 2\mathrm{Re}(\langle f, g\rangle) + \langle g, g\rangle \\ &\leq \langle f, f\rangle + 2|\langle f, g\rangle| + \langle g, g\rangle\end{aligned}$$

Now, using the CBS inequality, we have

$$\begin{aligned}\langle f + g, f + g\rangle &\leq \langle f, f\rangle + 2\langle f, f\rangle^{\frac{1}{2}}\langle g, g\rangle^{\frac{1}{2}} + \langle g, g\rangle \\ &\leq \left(\langle f, f\rangle^{\frac{1}{2}} + \langle g, g\rangle^{\frac{1}{2}}\right)^2.\end{aligned}$$

Which leads directly to the Minkowski inequality. ♣

Definition 3 (Square Integrable). *A function $f : \Re \to \mathcal{C}$ is called square integrable if the inner product exists and is finite, i.e.,*

$$\langle f, f\rangle = \int_{-\infty}^{+\infty} |f(t)|^2\ dt < \infty$$

For all practical purposes, all finite length signals are square integrable. In the signal processing literature, the square integrability contidion is often stated as the finite energy condition.

Definition 4 (Characteristic Function). *Let $\mathcal{S} \subset \Re$. The characteristic function of the set $\mathcal{S}$ is defined by:*

$$\chi_s(t) = \begin{cases} 1, & \text{if } t \in \mathcal{S} \\ 0, & \text{if } t \notin \mathcal{S} \end{cases}$$

Definition 5 (Measure). *The measure of a set $\mathcal{S} \subset \Re$ is defined by the integral*

$$\int_{-\infty}^{+\infty} \chi_s(t)\ dt$$

of its characteristic function.

Definition 6 (Almost everywhere). *Let f, g be two functions. If the set of points*

$$\{t \in \Re : f(t) \neq g(t)\}$$

has measure 0, then we say $f = g$, almost everywhere (a.e.).

Definition 7 (Convolution). *For any two functions $f, g \in \mathcal{L}^2(\Re)$, the convolution of f and g is defined by*

$$f * g = \int_{-\infty}^{+\infty} f(u)g(t-u)\ du$$

Theorem 4 ($\mathcal{L}^2(\Re)$). *If $\mathcal{L}^2(\Re)$ is the set defined by*

$$\{f : \Re \rightarrow \mathcal{C} : \int_{-\infty}^{+\infty} |f(t)|^2\ dt < \infty\},$$

i.e., the set of all complex–valued functions of one real variable that are square itegrable. Any two functions $f, g \in \mathcal{L}^2(\Re)$ that are equal a.e. are considered to be the same element of $\mathcal{L}^2(\Re)$. $\mathcal{L}^2(\Re)$ is an inner product space.

Proof: For a proof that $\mathcal{L}^2(\Re)$ is an inner product space, see page 408 of Marsden (1974). ♣

Note that the definition of $\mathcal{L}^2(\Re)$ can be generalized to $\Re^n$, though we will only consider the real line in what follows.

Theorem 5 (Properties of $\mathcal{L}^2(\Re)$). *If $f, g \in \mathcal{L}^2(\Re)$, then*

1. $f + g \in \mathcal{L}^2(\Re)$
2. $fg \in \mathcal{L}^2(\Re)$
3. $f * f \in \mathcal{L}^2(\Re)$, *where $*$ represents the convolution operator.*

Proof: Use the CBS and Minkowski inequalities. To prove 3 we need the Plancherel theorem (to be discussed later). ♣

Definition 8 (Fourier Transform). *The Fourier transform (FT) of a square integrable function $f \in \mathcal{L}^2(\Re)$ is written as*

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \langle e^{i\omega t}, f \rangle$$

Definition 9 (Inverse FT). *For a square integrable function $\hat{f} \in \mathcal{L}^2(\Re)$, the inverse FT (IFT) is defined by*

$$f(t) = \frac{1}{\sqrt{2\pi}} \langle \hat{f}^*, e^{i\omega t} \rangle$$

Theorem 6 (Plancherel Theorem). *The IFT of the FT of any $f \in \mathcal{L}^2(\Re)$ is equal to the original function a.e.*

Proof: The proof is in Marsden (1974) ♣

Theorem 7 (Orthogonality of exponentials). *The inner product of exponentials is*

$$\langle e^{i\mu t}, e^{i\nu t}\rangle = 2\pi\delta(\nu-\mu)$$

where

$$\delta(\nu-\mu) = \begin{cases} 0, & \text{if } \nu \neq \mu \\ 1, & \text{if } \nu = \mu \end{cases}$$

is the Dirac delta function.

The existence of this function was vehemently denied by mathematicians after its introduction. Some people still argue about it nowadays, not being aware of the work of Lighthill(1957).

Theorem 8 (Parsevals Theorem). *For* $f, g \in \mathcal{L}^2(\Re)$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

Theorem 9 (FT's of derivatives). *This is stated as*

$$\langle e^{i\omega t}, D^n f\rangle = (i\omega)^n \langle e^{i\omega t}, f\rangle$$

Theorem 9.5 (FT of a translated sequence). *The fourier transform of a translated sequence is a modulated version of the fourier transform of the original function or*

$$\langle e^{i\omega t}, f(x-x_0)\rangle = \exp -i\omega x_0 \langle e^{i\omega t}, f(x)\rangle$$

Proofs: Proofs of theorems 7, 8, 9 and 9.5 are left as exercises for the reader. ♣

In the chapter on multiresolution analysis we shall have occasion to make repeated use of the above theorems and in particular the FT of a translated sequence and the Parseval's relationship will be found to be most useful.

The uncertainty relation

Gabor's uncertainty principle places limits on the representation of any signal. Our derivation will make use of techniques discussed in the previous section and the following definitions.‡

Definition 9 (Hermitian Operators). *An operator* $A \in \mathcal{L}^2(\Re)$ *is Hermitian (or self–adjoint or symmetric) if, for* $f, g \in \mathcal{L}^2(\Re)$*, we have*

$$\langle f, Ag\rangle = \langle Af, g\rangle$$

Example

In this example, we show that the operator $iD = i\frac{d}{dt}$ is Hermitian. Note, that unless otherwise stated, all limits of integrals are from $-\infty$ to $+\infty$. Starting from the definition we have,

$$\begin{aligned} \langle f, iDg\rangle &= \int_{-\infty}^{+\infty} f^* iDg \;\; dt \\ 0 &= \int iD[f^*g] \;\; dt - \int [iDf^*]g \;\; dt \\ &= -\int [iDf^*]g \;\; dt \\ &= \int [iDf]^* g \;\; dt \\ &= \langle iDf, g\rangle \end{aligned}$$

which proves our result.

‡ The *other* uncertainty principle has been jokingly defined as – "Heisenberg may have been here."

Definition 10 (Expectation value). *For operator $A \in \mathcal{L}^2(\Re)$ and function $f \in \mathcal{L}^2(\Re)$, the expectation value of A is*

$$\langle A \rangle_f = \frac{\langle f, Af \rangle}{\langle f, f \rangle}$$

By way of explanation, consider a signal f, for example the chiming of Big Ben. when did it strike twelve?† This is not an easy question to answer, since no matter how short the sound, it occupies a finite time interval. A useful definition of time of occurrence is:

$$\begin{aligned} \langle t \rangle_f &= \frac{\int t |f|^*}{\int |f|^*} \\ &= \frac{\langle f, tf \rangle}{\langle f, f \rangle} \end{aligned}$$

In many signal processing texts, this definition is introduced without any notions of Hermitian operators.

Definition 11 (Variance). *The variance of an operator $A \in \mathcal{L}^2(\Re)$ whose expectation $\langle A \rangle_f$ is given by*

$$\Delta A^2 = \left\langle (A - \langle A \rangle_f)^2 \right\rangle_f$$

Theorem 10 (Variance). *For a linear operator $A \in \mathcal{L}^2(\Re)$ and for a function $f \in \mathcal{L}^2(\Re)$, the variance of A is given by*

$$\left\langle (A - \langle A \rangle_f)^2 \right\rangle_f = \left\langle A^2 \right\rangle_f - \left\langle A \right\rangle_f^2$$

Proof:

$$\begin{aligned} \left\langle (A - \langle A \rangle_f)^2 \right\rangle_f \langle f, f \rangle &= \int f^* (A - \langle A \rangle_f)^2 f \; dt \\ &= \int f^* (A^2 - A\langle A \rangle_f - \langle A \rangle_f A + \langle A \rangle_f^2 f \; dt \\ &= (\langle A^2 \rangle_f - 2\langle A \rangle_f \langle A \rangle_f + \langle A \rangle_f^2) \langle f, f \rangle \\ &= (\langle A^2 \rangle_f - \langle A \rangle_f^2) \langle f, f \rangle \end{aligned}$$

♣

Theorem 11 (Expectation of Hermitian Operators). *The expectation value of an Hermitian operator is real, i.e., $\langle H \rangle_f \in \Re$ for any $f \in \mathcal{L}^2(\Re)$.*

Proof: By the definition of the inner product $\langle f, Hf \rangle^* = \langle Hf, f \rangle$. However, H is Hermitian, therefore $\langle Hf, f \rangle = \langle f, Hf \rangle$ or $\langle f, Hf \rangle^* = \langle f, Hf \rangle$. Hence $\langle H \rangle_f$ is real. ♣

Theorem 12 (More on Hermitian Operators). *If H is an Hermitian operator, then $(H - \langle H \rangle_f)$ is Hermitian.*

Proof: We start with

$$\begin{aligned} \langle f, (H - \langle H \rangle_f) g \rangle &= \int f^* (H - \langle H \rangle_f) g \; dt \\ &= \int f^* H g \; dt - \int f^* \langle H \rangle_f g \; dt \\ &= \int ([Hf]^* g) - \int (\langle H \rangle_f f)^* g \; dt \\ &= \langle Hf, g \rangle - \langle \langle H \rangle_f f, g \rangle \\ &= \langle (H - \langle H \rangle_f) f, g \rangle. \end{aligned}$$

† It probably occurred when the mouse ran down.

♣

Theorem 13 (General Uncertainty Principle). *Let H and K be two Hermitian operators and let $f \in \mathcal{L}^2(\Re)$ be any square integrable function. Then the product of the uncertainties in H and K satisfy*

$$\Delta H \Delta K \geq \frac{1}{2}|\langle HK - KH\rangle_f|$$

Proof: Define $g = (H - \langle H\rangle_f)f,\ h = (K - \langle K\rangle_f)f$. Hence we can write

$$\langle g, g\rangle_f = \langle (H - \langle H\rangle_f)f, (H - \langle H\rangle_f)f\rangle_f.$$

Now, $(H - \langle H\rangle_f)$ is Hermitian, so we get:

$$\begin{aligned}\langle g, g\rangle_f &= \langle f, (H - \langle H\rangle_f)^2 f\rangle \\ &= \langle (H - \langle H\rangle_f)^2\rangle_f \langle f, f\rangle \\ &= \Delta H^2 \langle f, f\rangle.\end{aligned}$$

Similarly, $\langle h, h\rangle = \Delta K^2\langle f, f\rangle$. $\langle g, h\rangle$ is obtained in the following manner:

$$\begin{aligned}\langle g, h\rangle &= \langle (H - \langle H\rangle_f)f, (K - \langle K\rangle_f)f\rangle \\ &= \langle f, (H - \langle H\rangle_f)(K - \langle K\rangle_f)f\rangle \\ &= \langle f, HKf\rangle - \langle H\rangle_f\langle f, Kf\rangle - \langle K\rangle_f\langle f, Hf\rangle - \langle H\rangle_f\langle K\rangle_f\langle f, f\rangle\end{aligned}$$

But,

$$\langle H\rangle_f\langle f, Kf\rangle = \frac{\langle f, Hf\rangle}{\langle f, f\rangle}\langle f, Kf\rangle = \langle K\rangle_f\langle f, Hf\rangle$$

hence,

$$\begin{aligned}\langle g, h\rangle &= \langle f, HKf\rangle - \langle H\rangle_f\langle K\rangle_f\langle f, f\rangle \\ &= \big(\langle HK\rangle_f - \langle H\rangle_f\big)\langle f, f\rangle.\end{aligned}$$

Now use the CBS inequality $|\langle g, h\rangle|^2 \leq \langle g, g\rangle\langle h, h\rangle$, to get

$$|\big(\langle HK\rangle_f - \langle H\rangle_f\big)\langle f, f\rangle|^2 \leq \Delta H^2\Delta K^2\langle f, f\rangle^2$$

We are now close to the general form of the uncertainty principle. In the left hand side of the previous equation, we see that the terms $\langle . \rangle_f$ are real numbers, while the term $\langle HK\rangle_f$ is in general complex. So the left hand side is complex. Now write $HK = \frac{1}{2}\big[(HK + KH) + (HK - KH)\big]$. So,

$$\begin{aligned}\langle HK\rangle_f &= \frac{1}{2}\langle (HK + KH) + (HK - KH)\rangle_f \\ &= \frac{1}{2}\Big|\frac{\langle f, (HK + KH)f\rangle + \langle f, (HK - KH)f\rangle}{\langle f, f\rangle}\Big|\end{aligned}$$

The first term on the right hand side can be simplified as follows:

$$\begin{aligned}\langle f, (HK + KH)f\rangle &= \langle f, HKf\rangle + \langle f, KHf\rangle \\ &= \langle Hf, Kf\rangle + \langle Kf, Hf\rangle \\ &= \langle Hf, Kf\rangle + \langle Hf, Kf\rangle^* \\ &= a + ib + a - ib\end{aligned}$$

i.e., $\langle f, (HK+KH)f\rangle$ is a real number. Similarly

$$\langle f, (HK-KH)f\rangle = \langle Hf, Kf\rangle - \langle Hf, Kf\rangle^*$$

which is an imaginary number. Finally,

$$\begin{aligned} |(\langle HK\rangle_f - \langle H\rangle_f)\langle f, f\rangle|^2 &\geq \frac{1}{4}|\langle HK-KH\rangle_f|^2 \\ \Delta H^2 \Delta K^2 &\geq \frac{1}{4}|\langle HK-KH\rangle_f|^2 \end{aligned}$$

♣

Theorem 14 (Gabor's uncertainty principle). *For any function $f \in \mathcal{L}^2(\Re)$, the product of its uncertainty in time and frequency is bounded below by $\frac{1}{2}$.*

Proof: There are two parts to the proof:

1. We prove that the Hermitian operator iD has an expectation value of $\langle\omega\rangle_{\hat{f}}$.
2. We use the general uncertainty principle to get the desired inequality.

$$\langle iD\rangle_f = \frac{\langle f, -iDf\rangle}{ff}$$

Use Parsevals theorem to get

$$\begin{aligned} \langle iD\rangle_f &= \frac{\langle \hat{f}, \mathcal{F}[-iDf]\rangle}{\langle \hat{f}, \hat{f}\rangle} \\ &= \frac{\langle \hat{f}, -ii\omega\hat{f}\rangle}{\langle \hat{f}, \hat{f}\rangle} \\ &= \frac{\langle \hat{f}, \omega\hat{f}\rangle}{\langle \hat{f}, \hat{f}\rangle} \\ &= \langle\omega\rangle_{\hat{f}} \end{aligned}$$

For the second part of the proof use $H = t$ and $K = -iD$ and plug into the general uncertainty principle:

$$\Delta t \Delta\omega = \frac{1}{2}|\langle t(-iD) - (iD)t\rangle_f|$$

The term in $|.|$ is simplifed as follows:

$$\begin{aligned} \langle t(-iD) - (iD)t\rangle_f &= \frac{\langle f, -itDf\rangle - \langle f, -ifDt\rangle - \langle f, -itDf\rangle}{\langle f, f\rangle} \\ &= \frac{-\langle f, -ifDt\rangle}{\langle f, f\rangle} \\ &= \frac{i\langle f, f\rangle}{\langle f, f\rangle} \end{aligned}$$

which leads directly to the desired result.

Properties of Gaussians. The Gaussian is a fundamental function in all of the sciences and is written as:

$$\mathcal{N}_{\mu,\sigma}(x) = \frac{\exp\frac{(x-\mu)^2}{2\sigma^2}}{\sqrt{2\pi}\sigma}$$

For an interesting history of its statistical applications see Stigler, S. M (1974). Below we present some of its properties:

1. $\int_{-\infty}^{+\infty} \mathcal{N}_{\tau,\sigma}(t)\ dt = 1$
2. $\int_{-\infty}^{+\infty} t\mathcal{N}_{\tau,\sigma}(t)\ dt = \tau$
3. $\int_{-\infty}^{+\infty} (t-\tau)^2\mathcal{N}_{\tau,\sigma}(t)\ dt = \sigma^2$
4. $\mathcal{F}(\mathcal{N}_{0,\sigma}(\omega) = \frac{1}{\sigma}\mathcal{N}_{\frac{0,1}{\sigma}}(\omega)$

Proof of Minimum Uncertainty. Choose $f \in \mathcal{L}^2(\Re)$ for which $|f|^2$ is $\mathcal{N}_{0,\sigma}(t)$ or

$$\begin{aligned} f_{\sigma,\mu}(t) &= \sqrt{\mathcal{N}_{0,\sigma}(t)}\exp i\mu t \\ &= \exp i\mu t(8\pi)^{1/4}\sqrt{\sigma}\mathcal{N}_{0,\sqrt{2}\sigma}(t) \end{aligned}$$

With this choice of $f_{\sigma,\mu}$ we can write $|f_{\sigma,\mu}(t)|^2 = \mathcal{N}_{0,\sigma}(t)$. Now using the properties of the gaussian we have:

$$\Delta t^2 = \left\langle (t - \langle t\rangle_f)^2\right\rangle_{f_{\sigma,\mu}} = \sigma^2$$

To compute $\Delta\omega^2$ we take the fourier transform of $f_{\sigma,\mu}$ and doing our calculations in the frequency domain. We also use the properties of the gaussian discussed earlier. The FT of $\sqrt{\mathcal{N}_{0,\sigma}(t)}$ is

$$\hat{f}_{\sigma,0} = \mathcal{F}\left[\sqrt{\mathcal{N}_{0,\sigma}(t)}\right] = (\frac{2}{\pi})^{\frac{1}{4}}\sqrt{\sigma}\exp -\omega^2\sigma^2$$

or

$$\hat{f}_{\sigma,0} = \frac{(2\pi)^{\frac{1}{4}}}{\sqrt{\sigma}}\mathcal{N}_{0,(\sigma\sqrt{2})^{-1}}(\omega)$$

For $\mu \neq 0$, the FT of $f_{\sigma,\mu}(t)$ is

$$\hat{f}_{\sigma,\mu} = \frac{1}{\sqrt{2\pi}}\langle \exp -i\omega t, f_{\sigma,\mu}\rangle$$

Using our earlier definitions for $f_{\sigma,\mu}$ and $f_{\sigma,0}$ we can write

$$\begin{aligned} \hat{f}_{\sigma,\mu} &= \frac{1}{\sqrt{2\pi}}\langle \exp -i\omega t, \exp i\mu t f_{\sigma,0}\rangle \\ &= \frac{1}{\sqrt{2\pi}}\langle \exp -i(\omega-\mu)t, f_{\sigma,0}\rangle \end{aligned}$$

Therefore, $\hat{f}_{\sigma,\mu}(\omega) = \hat{f}_{\sigma,0}(\omega-\mu)$. Hence we write

$$\begin{aligned} |\hat{f}_{\sigma,0}(\omega-\mu)|^2 &= |\frac{(2\pi)^{\frac{1}{4}}}{\sqrt{\sigma}}\mathcal{N}_{\mu,(\sigma\sqrt{2})^{-1}}(\omega)|^2 \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}}2\sigma\exp\{\frac{-(\omega-\mu)^2}{(\frac{1}{sqrt2\sigma})^2}\} \\ &= \mathcal{N}_{\frac{\mu,1}{2\sigma}}(\omega) \end{aligned}$$

Now,

$$\begin{aligned} \langle\omega\rangle_{\hat{f}_{\sigma,\mu}} &= \frac{\int \omega|\hat{f}_{\sigma,0}(\omega-\mu)|^2\ d\omega}{\int |\hat{f}_{\sigma,0}(\omega-\mu)|^2\ d\omega} \\ &= \frac{\int \omega\mathcal{N}_{\frac{\mu,1}{2\sigma}}(\omega)\ d\omega}{\int \mathcal{N}_{\frac{\mu,1}{2\sigma}}(\omega)d\omega} \end{aligned}$$

Therefore, $\langle \omega \rangle_{\hat{f}_{\sigma,\mu}} = \mu$ and similarly,

$$\begin{aligned}
\Delta\omega^2 &= \langle (\omega - \langle \omega \rangle_{\hat{f}}) \rangle_{\hat{f}_{\sigma,\mu}} \\
&= \int (\omega - \mu)^2 \mathcal{N}_{\mu, \frac{1}{2\sigma}}(\omega)\ d\omega \\
&= (\frac{1}{2\sigma})^2
\end{aligned}$$

thus proving that the Gaussian achieves the lower bound for the uncertainty principle.

With these derivations and expositions out of the way, we are in a position to approach the continuous wavelet transform in the next chapter.